SAAS 91. Dynamics of aerial robots. Mechanical and Mathematical Modeling.

Jean Lerbet
http://www.personnels.univ-evry.fr/jeanlerbet/

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Introduction

In this course, we plan to provide to students, who are learning how to control airplanes, airships or any spatial system or spatial robot, the foundations of the mechanical modeling of such engines. They are often modeled by a single rigid body without any kinematical constraint so that they are mainly full 6 d.o.f. rigid bodies. Consequently, we focus here on the dynamics of a single rigid body. We complete this course by giving some elements about systems of rigid bodies. It could be applied for example to a satellite deployment. In the first part, we recall the essential mathematical tools. In the two next sections, the kinematics and the kinetics of a rigid body are presented. In the fourth section, we are concerned by the dynamics. The fifth part focuses on specific features of dynamics of flying objects and their control although other specific features of dynamics of space objects will not be tackled: fluid-structure interactions, powertrain issues,.... The sixth part is devoted to change of frames. The next section presents the extension to systems of rigid bodies. In the eighth and last section a specific attention is paid to vibrations of systems and the application to stability. Remark that a more condensed manner using Lie group theory could be used for such a presentation but it necessitates then a too significant mathematical background. Here, only matrices calculation and elementary differential calculus are supposed known. The other part of the teaching called Aerial Robots gives the usual terms of modelings of Aerial Robots when the control of these objects is the main goal (stabilization, path planing,....) of the model. In our part, we give the principles of mechanic laws and the mechanical meaning of the terms appearing in the equations. In this sense, both parts are complementary.

To conclude this introduction, it is worth mentioning that, very simple mechanical systems like a planar bi-pendulum or triple pendulum may exhibit a chaotic behavior (see for example this enlightening simulation `https://jakevdp.github.io/blog/2017/03/08/triple-pendulum-chaos` where 41 triple pendulums with slightly different initial conditions exhibit quickly largely different trajectories). The reason is the fundamental non linearities of the dynamic equations of any mechanical system. The consequence is that, after having derived the equations of the model (which is the main purpose of this course), only two options are really possible for using these equations. The first one is to control these systems and the second one is to investigate the linearization of these equations with respect to an equilibrium or a reference trajectory. Such investigations may be useful for vibration analysis. In the last part of this course, we give some elements for linear vibration analysis of mechanical systems.

1 Mathematical tools

1.1 Basic notions and notations.

$\mathcal{E}$ is the canonical 3 dimensional euclidean affine space and E is its canonically associated vector space with $+$ as internal law. To represent elementary objects of E called vectors we use bases that will always be used orthonormal bases. It means that for such a basis $\mathcal{B} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$, then $(\vec{e}_i \mid \vec{e}_j) = \delta_{ij}$ where $(. \mid .)$ is the usual scalar product of E. If $\vec{x} \in E$, $\vec{x} = \sum_{i=1}^{3} x_i \vec{e}_i$ and (x_1, x_2, x_3) are the coordinates of $\vec{x}$ in $\mathcal{B}$. $\wedge$ is the usual cross product of E. An orthonormal basis $\mathcal{B} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is direct iff $(\vec{e}_3) = \vec{e}_1 \wedge \vec{e}_2$ and then $x_i = (\vec{x} \mid \vec{e}_i)$

for all i.
Elementary objects of $\mathcal{E}$ are called points and are noted by capital letters like $O, M, N, \ldots$. and if $M \in \mathcal{E}$ the unique vector $\vec{v}$ such that $M = N + \vec{v}$ is denoted $\overrightarrow{NM}$.
To represent points, we use coordinate systems $\mathcal{F} = (O; \mathcal{B})$ built from any element $O \in \mathcal{E}$ and any basis $\mathcal{B}$. For any $M \in \mathcal{E}$, $M = O + \sum_{i=1}^{3} X_{M,i}\vec{e}_i = O + \sum_{i=1}^{3} X_i\vec{e}_i$ and $(X_{M,1}, X_{M,2}, X_{M,3}) = (X_1, X_2, X_3)$ are the coordinates of M in $\mathcal{F}$.
Because this curse is built in the framework of classical mechanics and not of relativity, we may suppose that the time is absolute and often denoted by t. In this case a frame allowing to observe a motion of points in $\mathcal{E}$ is equivalent to a coordinate system.

1.2 Rotations of E and Euclidean displacements of $\mathcal{E}$.

First of all, some definitions. A map $A : \mathcal{E} \to \mathcal{E}$ is an affine map if there is a linear map $u = u_A \in \mathcal{L}(E)$ such that for all $M, P \in \mathcal{E}$, $A(M) = A(N) + u_A(\overrightarrow{NM})$. u_A is called the linear part of A. A linear map $u \in \mathcal{L}(E)$ is an orthogonal map or an orthogonal transformation if $(u(\vec{x}) \mid u(\vec{y})) = (\vec{x} \mid \vec{y})$ for all $\vec{x}, \vec{y} \in E$. Recall that an element $u \in \mathcal{L}(E)$ is symmetric ($\in \mathcal{S}(E)$) if $(u(\vec{x}) \mid \vec{y}) = (\vec{x} \mid u(\vec{y}))$ for all $\vec{x}, \vec{y} \in E$ and is skew symmetric ($\in \mathcal{A}(E)$) if $(u(\vec{x}) \mid \vec{y}) = -(\vec{x} \mid u(\vec{y}))$ for all $\vec{x}, \vec{y} \in E$.
The image $u(\mathcal{B}) = (u(\vec{e}_1), u(\vec{e}_2), u(\vec{e}_3))$ of an orthonormal basis $\mathcal{B} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ by an orthogonal map is still an orthonormal basis (that can be used as a definition of an orthogonal transformation!). The set of all orthogonal transformations is noted $\mathcal{O}(E)$.
If an orthonormal basis $\mathcal{B} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is given, the matrix of any linear map $u \in \mathcal{L}(E)$ is a 3×3 matrix $\mathbf{M}_{\mathcal{B}}(u) \in \mathcal{M}_3(\mathbb{R})$ with $\mathbf{M}_{\mathcal{B}}(u)_{ij} = (u(e_j) \mid e_i)\ \forall i, j$. $u \in \mathcal{O}(E)$ if and only if $\mathbf{M}_{\mathcal{B}}(u)$ is an orthogonal matrix meaning that $\mathbf{M}_{\mathcal{B}}(u) \in O_3(\mathbb{R}) = \{A \in \mathcal{M}_3(\mathbb{R}) \mid A^T A = AA^T = \mathrm{I}_3\}$. It is a group for the composition as internal law. The subset (subgroup) of elements $u \in \mathcal{O}(E)$ with $\det(u) = 1$ is the special orthogonal group of E denoted $\mathcal{SO}(E)$. It is a 3 dimensional Lie group. Its Lie algebra is the 3 dimensional vector subspace $\mathcal{A}(E)$ of $\mathcal{L}(E)$. Because any skew symmetric map $u \in \mathcal{A}(E)$ may be represented by the cross product by a vector ω_u ($u(\vec{x}) = \omega_u \wedge \vec{x}$ for all $\vec{x} \in E$), this Lie algebra may be identified with the Lie algebra E equipped with its cross product $\wedge$. It is a good exercise to prove the Jacobi's identity:

$$\forall \vec{x}, \vec{y}, \vec{z} \in E \quad \vec{x} \wedge (\vec{y} \wedge \vec{z}) + \vec{y} \wedge (\vec{z} \wedge \vec{x}) + \vec{z} \wedge (\vec{x} \wedge \vec{y}) = \vec{0} \tag{1}$$

without forgetting that $\vec{x} \wedge (\vec{y} \wedge \vec{z}) = (\vec{x} \mid \vec{z})\vec{y} - (\vec{x} \mid \vec{y})\vec{z}\ \ \forall \vec{x}, \vec{y}, \vec{z} \in E$.
Introducing the adjoint operator u^* of u, we have the following characterizations:

$$u \in \mathcal{SO}(E) \Longleftrightarrow u \circ u^* = \mathrm{Id}_E, u \in \mathcal{S}(E) \Longleftrightarrow u = u^*, u \in \mathcal{A}(E) \Longleftrightarrow u = -u^*$$

Again $u \in \mathcal{S}(E)$ if and only if $\mathbf{M}_{\mathcal{B}}(u)$ is a symmetric matrix meaning that $\mathbf{M}_{\mathcal{B}}(u)^T = \mathbf{M}_{\mathcal{B}}(u)$ and $u \in \mathcal{A}(E)$ if and only if $\mathbf{M}_{\mathcal{B}}(u)$ is a skew symmetric matrix meaning that $\mathbf{M}_{\mathcal{B}}(u)^T = -\mathbf{M}_{\mathcal{B}}(u)$. These assertions are true if and only if $\mathcal{B} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is orthonormal.
Let now $t \mapsto u(t)$ be a differentiable map from $]-1, 1[$ to $\mathcal{SO}(E)$ with $u(0) = \mathrm{Id}_E$. Then for all $t \in]-1, 1[$, $u(t) \circ u^*(t) = \mathrm{Id}_E$. Differentiating this relation gives: $u'(t) \circ u^*(t) + u(t) \circ {u^*}'(t) = 0_E$. Taking the value for $t = 0$ leads to $u'(0) + {u^*}'(0) = 0_E$ meaning that $u'(0) = -{u^*}'(0)$ or $u'(0) \in \mathcal{A}(E)$. To this element is canonically

associated a vector $\vec{\omega}_u(0) = \vec{\omega}_{u'(0)} \in E$. More generally, $u'(t) \circ u^*(t) = -u(t) \circ u^{*'}(t) = -(u'(t) \circ u^*(t))^*$ leads to two elements of $\mathcal{A}(E)$ and consequently to two vectors of E:

- the left or Lagrangian representation of $u'(t)$: $\vec{\Omega}_{u'(t)} = \vec{\omega}_{u^*(t) \circ u'(t)}$
- the right or Eulerian representation of $u'(t)$: $\vec{\omega}_{u'(t)} = \vec{\omega}_{u'(t) \circ u^*(t)}$

An affine map A is an Euclidean displacement iff its linear part u_A belongs to $\mathcal{SO}(E)$. It is equivalent to suppose that A preserves the orientation of $\mathcal{E}$ and the distance between two any points $M, N \in \mathcal{E}$: $d(A(M), A(N) = d(M, N)$. The set $\mathbb{D}(\mathcal{E})$ of Euclidean displacements is a 6 dimensional Lie group and its Lie algebra may be identified with the vector space $\mathfrak{D}(\mathcal{E})$ of skew-symmetric vector fields on $\mathcal{E}$. A more detailed study of this set is reported hereafter.

1.3 The Lie algebra $\mathfrak{D}(\mathcal{E})$

Let $\mathcal{A}(\mathcal{E}, E)$ the set of affine vector fields on $\mathcal{E}$ meaning the set of $X : \mathcal{E} \to E$ such that there is $u = u_X \in \mathcal{L}(E)$ such that for all $M, P \in \mathcal{E}$, $X(M) = X(P) + u_X(\overrightarrow{PM})$. Recall that $\mathfrak{D}(\mathcal{E}) = \{X \in \mathcal{A}(\mathcal{E}, E) \mid u_X \in \mathcal{A}(E)\}$. So, for all $X \in \mathfrak{D}(\mathcal{E})$, there is a unique $\omega_X \in E$ such that

$$X(M) = X(P) + \omega_X \wedge \overrightarrow{PM} \quad \forall M, P \in \mathcal{E} \tag{2}$$

Elements of $\mathfrak{D}(\mathcal{E})$ are called skew symmetric fields (s.s.f.). It is a Lie algebra for the Lie bracket defined by:

$$[X, Y](M) = \omega_X \wedge Y(M) - \omega_Y \wedge X(M) \quad \forall M \in \mathcal{E} \tag{3}$$

It is a good exercise to prove that $\mathfrak{D}(\mathcal{E})$ equipped with $[., .]$ is a 6 dimensional Lie algebra such that $\omega_{[X,Y]} = \omega_X \wedge \omega_Y$ for all $X, Y \in \mathfrak{D}(\mathcal{E})$. If $\mathcal{F} = (O; \mathcal{B})$ is an orthonormal coordinate system of $\mathcal{E}$, we built a basis $(X_1, \ldots, X_6)$ of $\mathfrak{D}(\mathcal{E})$ by $X_1(M) = \vec{e}_1, X_2(M) = \vec{e}_2, X_3(M) = \vec{e}_3), X_4(M) = \vec{e}_1 \wedge \overrightarrow{OM}, X_5(M) = \vec{e}_2 \wedge \overrightarrow{OM}, X_6(M) = \vec{e}_3 \wedge \overrightarrow{OM})$ for all $M \in \mathcal{E}$.

1.4 Bases and frames changing.

Let $\mathcal{B}$ and $\mathcal{B}'$ two orthonormal bases of E. At each vector $\vec{x} \in E$, is associated the column vector ${}^T[x_1\ x_2\ x_3] = col(\vec{x}, \mathcal{B}) = \mathbf{x}_\mathcal{B} = \mathbf{x}$ (if no confusion is possible) of coordinates of $\vec{x}$ in $\mathcal{B}$. By the same way, ${}^T[x'_1 x'_2 x'_3] = \mathbf{x}_{\mathcal{B}'} = col(\vec{x}, \mathcal{B}') = \mathbf{x}'$ (in no confusion is possible) is the column vector of coordinates of $\vec{x}$ in $\mathcal{B}'$. Let $P_{\mathcal{B},\mathcal{B}'} = \mathbf{P}$ the matrix of components of vectors of $\mathcal{B}'$ in $\mathcal{B}$ allowing to pass from $\mathcal{B}$ to $\mathcal{B}'$. $P_{\mathcal{B},\mathcal{B}'} = \mathbf{P}$ is called a change of bases matrix (c.b.m.) and because both the bases are direct and orthonormal, $P_{\mathcal{B},\mathcal{B}'} = \mathbf{P}$ is an element of the special orthogonal group $SO_3(\mathbb{R}) = \{A \in O_3(\mathbb{R}) \mid \det A = 1\}$.. Be careful that

$$\mathbf{x} = \mathbf{P}\mathbf{x}' \tag{4}$$

holds and not the opposite relationship. If $\mathcal{B}''$ is a third orthonormal basis the following relationship holds:

$$P_{\mathcal{B},\mathcal{B}''} = P_{\mathcal{B},\mathcal{B}'} P_{\mathcal{B}',\mathcal{B}''} \tag{5}$$

Similar transformations exist for linear maps $u \in \mathcal{L}(E)$. If $\mathbf{M}_\mathcal{B} = \mathbf{M}_\mathcal{B}(u) = \mathrm{Mat}(u, \mathcal{B}) = \mathbf{M}$ (if no confusion is possible) is the matrix of u in $\mathcal{B}$ (remember that $\mathbf{M}_{ij} = (u(\vec{e}_i) \mid \vec{e}_j))$ because the orthonormal basis $\mathcal{B}$ and then $\mathbf{M}' = \ \mathbf{P}^T\mathbf{M}\mathbf{P}$.
For $u \in \mathcal{A}(E)$ a skew symmetric linear map, we know that there is a unique vector $\vec{\omega_u} \in E$ with $u(\vec{x}) = \vec{\omega_u} \wedge \vec{x}$ for all $\vec{x} \in E$. In an orthonormal basis $\mathcal{B}$, the matrix $\mathbf{M}_\mathcal{B}(u)$ is a skew symmetric 3×3 matrix like

$$\mathbf{M}_\mathcal{B}(u) = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

so that the column vector $col(\vec{\omega}_u, \mathcal{B}) = \omega_\mathbf{u}$ of $\vec{\omega}_u$ in $\mathcal{B}$ is exactly ${}^T[a\ b\ c] = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. We also will write this last matrix $\mathbf{M}_\mathcal{B}(\vec{\omega}_u \wedge)$.
It follows that for any $col(\vec{x}, \mathcal{B}) =^T [x_1\ x_2\ x_3] = \mathbf{x}_\mathcal{B} = \mathbf{x}$ column vector of coordinates of $\vec{x} \in$ in $\mathcal{B}$, the column vector $col(u(\vec{x}), \mathcal{B})$ of coordinates of $u(\vec{x}) = \vec{\omega_u} \wedge \vec{x}$ in $\mathcal{B}$ is given by the product:

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} bx_3 - cx_2 \\ cx_1 - ax_3 \\ ax_2 - bx_1 \end{pmatrix}$$

1.5 Angular velocity

Suppose now that $t \mapsto \vec{x}(t)$ is a motion in E. The derivative of $\vec{x}(t)$ with respect to the basis $\mathcal{B}$ denoted by $\left.\frac{d\vec{x}(t)}{dt}\right)_\mathcal{B}$ is the vector

$$\left.\frac{d\vec{x}(t)}{dt}\right)_\mathcal{B} = \sum_{i=1}^{3} \dot{x}_i \vec{e}_i$$

where $\vec{x}(t) = \sum_{i=1}^{3} x_i \vec{e}_i = \sum_{i=1}^{3} x_i(t)\vec{e}_i(t)$ and where $\dot{x}_i = \frac{dx_i(t)}{dt}$. In other words, $col(\left.\frac{d\vec{x}(t)}{dt}\right)_\mathcal{B}, \mathcal{B}) = [\dot{x}_1, \dot{x}_2, \dot{x}_3]^T$.
Let $\mathcal{B}'$ be another basis and $P_{\mathcal{B},\mathcal{B}'}(t) = \mathbf{P}(t)$ the corresponding c.b.m. Consider $u(t) \in \mathcal{SO}(E)$ the linear map (the rotation!!) with matrix $\mathbf{P}(t)$ in the basis $\mathcal{B}$: $\mathbf{P}(t) = \mathbf{M}_\mathcal{B}(u(t))$. The Eulerian representation $\vec{\omega}_{\mathcal{B}'/\mathcal{B}} = \vec{\omega}_{\mathcal{B}'/\mathcal{B}}(t)$ of $\dot{u}(t)$ is also called the instantaneous rotation velocity of $\mathcal{B}'$ with respect to $\mathcal{B}$. In a matricial representation, the coordinates $col(\vec{\omega}_{\mathcal{B}'/\mathcal{B}}, \mathcal{B})$ of $\vec{\omega}_{\mathcal{B}'/\mathcal{B}}$ in $\mathcal{B}$ are $\omega_{\mathbf{P}(t)}$ so that

$$\dot{\mathbf{P}}(t)\ \mathbf{P}(t)^T = \mathbf{M}_\mathcal{B}(\vec{\omega}_{\mathcal{B}'/\mathcal{B}} \wedge)$$

Calculations then give:

$$\left.\frac{d\vec{x}(t)}{dt}\right)_\mathcal{B} = \left.\frac{d\vec{x}(t)}{dt}\right)_{\mathcal{B}'} + \vec{\omega}_{\mathcal{B}'/\mathcal{B}} \wedge \vec{x}(t) \tag{6}$$

□ Proof: We have $\mathbf{x}_{\mathcal{B}}(t) = \mathbf{P}(t)\mathbf{x}_{\mathcal{B}'}(t)$ so that after a derivative:

$$\begin{aligned}
\dot{\mathbf{x}}_{\mathcal{B}}(t) &= \dot{\mathbf{P}}(t)\mathbf{x}_{\mathcal{B}'}(t) + \mathbf{P}(t)\dot{\mathbf{x}}_{\mathcal{B}'}(t) \\
&= \dot{\mathbf{P}}(t)\mathbf{P}^T(t)\mathbf{x}_{\mathcal{B}}(t) + \mathbf{P}(t)\dot{\mathbf{x}}_{\mathcal{B}'}(t) \text{ meaning that} \\
col(\left.\frac{d\vec{x}(t)}{dt}\right)_{\mathcal{B}}, \mathcal{B}) &= \mathbf{M}_{\mathcal{B}}(\vec{\omega}_{\mathcal{B}'/\mathcal{B}}\wedge)\mathbf{x}_{\mathcal{B}}(t) + col(\left.\frac{d\vec{x}(t)}{dt}\right)_{\mathcal{B}'}, \mathcal{B})
\end{aligned}$$

which is exactly the relation (6) in the basis $\mathcal{B}$. ∎

Definition 1 *The vector $\vec{\omega}_{\mathcal{B}'/\mathcal{B}}$ is called the angular velocity of $\mathcal{B}'$ with respect to $\mathcal{B}$. If $u \in \mathcal{SO}(E)$ is the rotation passing from $\mathcal{B}$ to $\mathcal{B}'$ (which is a function of t), $\vec{\omega}_{\mathcal{B}'/\mathcal{B}}$ is nothing else than the Eulerian representation of $u'(t)$.*

1.6 Composition of rotations and angular velocity

Suppose now that there are three (orthonormal) bases $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ and u and v are the two rotations so that (omitting the dependency with t) $\mathbf{P} = P_{\mathcal{B},\mathcal{B}'} = \mathbf{M}_{\mathcal{B}}(u)$, $\mathbf{P}' = P_{\mathcal{B}',\mathcal{B}''} = \mathbf{M}_{\mathcal{B}'}(v)$ and $\mathbf{P}'' = P_{\mathcal{B},\mathcal{B}''} = \mathbf{M}_{\mathcal{B}}(v \circ u)$. Then, it is well known that $\mathbf{P}'' = \mathbf{P}\mathbf{P}'$ meaning that

$$\mathbf{M}_{\mathcal{B}}(v \circ u) = \mathbf{M}_{\mathcal{B}}(v)\mathbf{M}_{\mathcal{B}}(u) = \mathbf{M}_{\mathcal{B}}(u)\mathbf{M}_{\mathcal{B}'}(v)$$

(BE CAREFUL TO THE ORDER of the operations). Then

Theorem 1

$$\vec{\omega}_{\mathcal{B}''/\mathcal{B}} = \vec{\omega}_{\mathcal{B}'/\mathcal{B}} + \vec{\omega}_{\mathcal{B}''/\mathcal{B}'} \tag{7}$$

□ Taking now the derivative of $\mathbf{P}'' = \mathbf{P}\mathbf{P}'$ gives $\dot{\mathbf{P}}'' = \dot{\mathbf{P}}\mathbf{P}' + \mathbf{P}\dot{\mathbf{P}}'$. Thus

$$\begin{aligned}
\dot{\mathbf{P}}''\mathbf{P}''^T &= (\dot{\mathbf{P}}\mathbf{P}' + \mathbf{P}\dot{\mathbf{P}}')(\mathbf{P}\mathbf{P}')^T \\
&= \dot{\mathbf{P}}\mathbf{P}'\mathbf{P}'^T\mathbf{P}^T + \mathbf{P}\dot{\mathbf{P}}'\mathbf{P}'^T\mathbf{P}^T \\
&= \dot{\mathbf{P}}\mathbf{P}^T + \mathbf{P}(\dot{\mathbf{P}}'\mathbf{P}'^T)\mathbf{P}^T \quad \text{because} \quad \mathbf{P}'\mathbf{P}'^T = \mathrm{I}_3 \text{ meaning that} \\
\mathbf{M}_{\mathcal{B}}(\vec{\omega}_{\mathcal{B}''/\mathcal{B}}\wedge) &= \mathbf{M}_{\mathcal{B}}(\vec{\omega}_{\mathcal{B}'/\mathcal{B}}\wedge) + \mathbf{P}\mathbf{M}_{\mathcal{B}'}(\vec{\omega}_{\mathcal{B}''/\mathcal{B}'}\wedge)\mathbf{P}^T \\
&= \mathbf{M}_{\mathcal{B}}(\vec{\omega}_{\mathcal{B}'/\mathcal{B}}\wedge) + \mathbf{M}_{\mathcal{B}}(\vec{\omega}_{\mathcal{B}''/\mathcal{B}'}\wedge) = \mathbf{M}_{\mathcal{B}}((\vec{\omega}_{\mathcal{B}'/\mathcal{B}} + \vec{\omega}_{\mathcal{B}''/\mathcal{B}'})\wedge)
\end{aligned}$$

leading to the result ∎

1.7 Summary: usual Lie groups and their Lie algebras in low dimensions

Lie group	Lie algebra	dimension	Lie bracket
$\mathbb{U} = \{z \in \mathbb{C}/\lvert z\rvert = 1\}$	$\mathbb{R}$	1	0 (trivial)
	E	3	$\wedge$
$\mathcal{O}(E)$	$\mathcal{A}(E)$	3	$u \circ v - v \circ u$
$\mathcal{SO}(E)$	$\mathcal{A}(E)$	3	$u \circ v - v \circ u$
$\mathcal{O}_3(\mathbb{R})$	$\mathcal{A}_3(\mathbb{R})$	3	$M_1M_2 - M_2M_1$
$\mathcal{SO}_3(\mathbb{R})$	$\mathcal{A}_3(\mathbb{R})$	3	$M_1M_2 - M_2M_1$
$\mathbb{D}(\mathcal{E})$	$\mathfrak{D}(\mathcal{E})$	6	$[X,Y](M) = \omega_X \wedge Y(M) - \omega_Y \wedge X(M)$
$\mathcal{G}l(E)$	$\mathcal{L}(E)$	9	$u \circ v - v \circ u$
$\mathcal{G}l_3(\mathbb{R})$	$\mathcal{M}_3(\mathbb{R})$	9	$M_1M_2 - M_2M_1$

1.8 Representations of rotations

There are infinite ways to represent any element u of $\mathcal{SO}(E)$ or any matrix $\mathbf{P} \in SO_3(\mathbb{R})$. In other words, there are infinite charts of $\mathcal{SO}(E)$ over each element u. Because $\mathcal{SO}(E)$ is a compact 3 dimensional manifold (namely it is a 3 dimensional Lie group), it cannot be covered by an unique chart. It means that a parametrization is always a local representation of $\mathcal{SO}(E)$ and that 3 independent parameters are necessary to describe any $u \in \mathcal{SO}(E)$. It also means that a fixed parametrization (chart) is always only local. Please then be careful that some values of parameters are singular and cannot more represent any rotation.
To well understand this problem, you may think to the circle

$$\mathbb{S}^1 = \mathbb{U} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} = \{z \in \mathbb{C}/\lvert z\rvert = 1\}$$

which is a one dimensional compact Lie group: it is obviously a one dimensional manifold and it is a group because it is also the set of complex numbers with modulus 1. The group multiplication law is then the multiplication of complex numbers: $(x,y)(x',y') = (xx' - yy', xy' + yx)$ and $(1,0)$ is the unity of the group. It is well-known that at least 2 charts are also necessary to cover the circle. For this case, the group is commutative, the Lie algebra is $\mathbb{R}$ and the Lie bracket is trivial namely nil.

1.8.1 Description by Axis $\vec{x}_u$ and angle θ_u

Geometrically speaking, any rotation u of $\mathcal{SO}(E)$ is defined by its axis Δ_u (or any vector $\vec{x}_u \neq 0 \in \Delta_u$ invariant under u: $u(\vec{x}_u) = \vec{x}_u$) and its angle θ_u. Choosing a normalized vector $\vec{x}_u$ leads to $2 + 1 = 3$ parameters. It is however not the usual way for parametrizing a rotation in a mechanical modeling. Indeed, there often are preferential directions in $\mathcal{E}$ or E in relationship with the body (like inertial axes or symmetric axes) or with the environment of the body (typically the gravity direction or the wind direction). It is the reason why the representation $u \in \mathcal{SO}(E)$ is often done as a product of three elementary successive rotations about suitable axes.
Remind however the well-known Olinde-Rodrigues formula:

$$u(\vec{x}) = \vec{x} + \sin\theta_u \vec{x}_u \wedge \vec{x} + (1 - \cos\theta_u)\vec{x}_u \wedge (\vec{x}_u \wedge \vec{x}) \qquad (8)$$

Choosing a direct orthonormal basis $\mathcal{B} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ such that $\vec{e}_1 \in \Delta_u$, then

$$\mathrm{Mat}(u, \mathcal{B}) = \mathbf{M}_1(\theta_u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_u & -\sin\theta_u \\ 0 & \sin\theta_u & \cos\theta_u \end{pmatrix}$$

If $\vec{e}_2 \in \Delta_u$, then

$$\mathrm{Mat}(u, \mathcal{B}) = \mathbf{M}_2(\theta_u) = \begin{pmatrix} \cos\theta_u & 0 & \sin\theta_u \\ 0 & 1 & 0 \\ -\sin\theta_u & 0 & \cos\theta_u \end{pmatrix}$$

and if $\vec{e}_3 \in \Delta_u$, then

$$\mathrm{Mat}(u, \mathcal{B}) = \mathbf{M}_3(\theta_u) = \begin{pmatrix} \cos\theta_u & -\sin\theta_u & 0 \\ \sin\theta_u & \cos\theta_u & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A straightforward calculation shows that the angular velocities associated with the above elementary rotations read successively $\dot{\theta}_u \vec{e}_1$, $\dot{\theta}_u \vec{e}_2$ and $\dot{\theta}_u \vec{e}_3$.
In the following paragraphs, we develop two usual representations, the second one being the most frequently used for modeling the dynamics of three dimensional flying objects. Recall once again that if $u, v \in \mathcal{L}(E)$ and if $\mathcal{B}$ is any basis of E, then $\mathrm{Mat}(v \circ u, \mathcal{B}) = \mathrm{Mat}(v, \mathcal{B})\mathrm{Mat}(u, \mathcal{B})$.

1.8.2 Description by Euler angles: precession ψ, nutation θ and own rotation ϕ

Let $\mathcal{B} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and $\mathcal{B}' = (\vec{e}_1', \vec{e}_2', \vec{e}_3')$ two direct orthonormal bases and $u \in \mathcal{SO}(E)$ so that $u(\mathcal{B}) = \mathcal{B}'$. Let $\mathbf{P} = P_{\mathcal{B},\mathcal{B}'} = \mathrm{Mat}(u, \mathcal{B})$. The three Euler angles are defined by the following way and in the following order:

- precession u_ψ of angle ψ about $\Delta_{\vec{e}_3}$
- nutation u_θ of angle θ about $\Delta_{u_\psi(\vec{e}_1)}$
- own rotation u_ϕ of angle ϕ about $\Delta_{u_\theta(\vec{e}_3)} = \Delta_{u_\theta \circ u_\psi(\vec{e}_3)}$

so that $u = u_\phi \circ u_\theta \circ u_\psi$. It is a good exercise to calculate $\mathbf{P}$ as function of parameters ψ, θ, ϕ and to prove that (be aware to the order of operations in comparison with $u = u_\phi \circ u_\theta \circ u_\psi$)

$$\mathbf{P} = \mathbf{M}_3(\psi)\mathbf{M}_1(\theta)\mathbf{M}_3(\phi)$$

□ A first direct proof uses the c.b.m. interpretation of involved matrices: $\mathbf{M}_3(\psi) = P_{\mathcal{B},u_\psi(\mathcal{B})}$, $\mathbf{M}_1(\theta) = P_{u_\psi(\mathcal{B}),u_\theta(u_\psi(\mathcal{B}))}$ and $\mathbf{M}_3(\phi) = P_{u_\theta(u_\psi(\mathcal{B})),u_\phi(u_\theta(u_\psi(\mathcal{B}))}$ and the main property (5) of c.b.m.
It is interesting to directly prove that $\mathbf{P} = \mathbf{M}_3(\psi)\mathbf{M}_1(\theta)\mathbf{M}_3(\phi) = \mathrm{Mat}(u, \mathcal{B})$. In order to use $u = u_\phi \circ u_\theta \circ u_\psi$ through matrix operations (multiplications), the matrices of u_ϕ, u_θ, u_ψ have to be evaluated in the same basis $\mathcal{B}$. We get successively: $\mathbf{M}_\mathcal{B}(u_\psi) = \mathbf{M}_3(\psi)$, $\mathbf{M}_\mathcal{B}(u_\theta) = P_{\mathcal{B},u_\psi(\mathcal{B})}\ \mathbf{M}_1(\theta) P^T_{\mathcal{B},u_\psi(\mathcal{B})} = \mathbf{M}_3(\psi)\ \mathbf{M}_1(\theta)\ \mathbf{M}_3(\psi)^T$ and

$$\begin{aligned} \mathbf{M}_\mathcal{B}(u_\phi) &= P_{\mathcal{B},u_\theta \circ u_\psi(\mathcal{B})}\ \mathbf{M}_3(\phi) P^T_{\mathcal{B},u\theta \circ u_\psi(\mathcal{B})} \\ &= (\mathbf{M}_3(\psi)\ \mathbf{M}_1(\theta))\ \mathbf{M}_3(\phi)\ (\mathbf{M}_3(\psi)\ \mathbf{M}_1(\theta)^T = \mathbf{M}_3(\psi)\ \mathbf{M}_1(\theta)\ \mathbf{M}_3(\phi)\ \mathbf{M}_1(\theta)^T\ \mathbf{M}_3(\psi)^T \end{aligned}$$

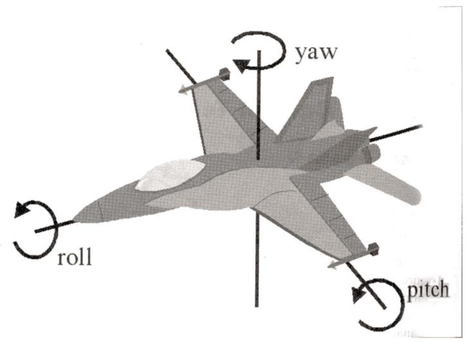

Figure 1: YawPitchRoll

and $\mathbf{P} = \mathbf{M}_{\mathcal{B}}(u_\phi \circ u_\theta \circ u_\psi) = \mathbf{M}_{\mathcal{B}}(u_\phi)\mathbf{M}_{\mathcal{B}}(u_\theta)\mathbf{M}_{\mathcal{B}}(u_\psi)$ leads to the wanted relation because of the orthogonality of each involved matrix.. ∎
Because of the previous theorem, the angular velocity attached to the Euler angles parametrization of a rotation reads:

$$\vec{\omega} = \dot{\psi}\vec{e}_3 + \dot{\theta}u_\psi(\vec{e}_1) + \dot{\phi}\vec{e}_3' = \dot{\psi}\vec{e}_3 + \dot{\theta}u_\psi(\vec{e}_1) + \dot{\phi}(u_\theta \circ u_\psi)(\vec{e}_3) = \dot{\psi}\vec{e}_3 + \dot{\theta}u_\psi(\vec{e}_1) + \dot{\phi}u_\theta(\vec{e}_3)$$

Evaluating the vectors $u_\psi(\vec{e}_1)$ and $u_\theta(\vec{e}_3)$ for example in the basis $\mathcal{B}$ gives:

$$\vec{\omega} = (\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\vec{e}_1 + (\dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi)\vec{e}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\vec{e}_3 \tag{9}$$

1.8.3 Description by Yaw ψ, pitch θ and roll ϕ angles

Let $\mathcal{B} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and $\mathcal{B}' = (\vec{e}_1', \vec{e}_2', \vec{e}_3')$ two direct orthonormal bases and $u \in \mathcal{SO}(E)$ so that $u(\mathcal{B}) = \mathcal{B}'$. Let $\mathbf{P} = P_{\mathcal{B},\mathcal{B}'} = \mathrm{Mat}(u, \mathcal{B})$. The three "Yaw-pitch-roll" angles are defined by the following way and in the following order:

- yaw u_ψ of angle ψ about $\Delta_{\vec{e}_3}$
- pitch u_θ of angle θ about $\Delta_{u_\psi(\vec{e}_2)}$
- roll u_ϕ of angle ϕ about $\Delta_{u_\theta \circ u_\psi(\vec{e}_1)}$

so that $u = u_\phi \circ u_\theta \circ u_\psi$. It is a good exercise to calculate $\mathbf{P}$ as function of parameters ψ, θ, ϕ and the angular velocity attached to this representation.

There is no universal definition of these angles excepted for small motions. In this case, the order of the successive three rotations is without significancy. For finite motion (here finite rotations), the chosen order is the above one: yaw (1), pitch (2),roll (3).

1.9 Duality

There is a last significant object in $\mathfrak{D}(\mathcal{E})$ called the inner product $[. \mid .]$ (or the comoment) that must not be confused with the Lie bracket $[.,.]$. If $X, Y \in \mathfrak{D}(\mathcal{E})$, $[X \mid Y] = (\omega_X \mid Y(M)) + (\omega_Y \mid X(M))$ where the right-hand side of the last relation is independent on $M \in \mathcal{E}$ and only depends on X and Y. $[. \mid .]$ is a symmetric non degenerate bilinear form on $\mathfrak{D}(\mathcal{E})$ but is not a scalar product on $\mathfrak{D}(\mathcal{E})$. Because of this form, any linear form $T \in \mathfrak{D}(\mathcal{E})^*$ called a torsor or a wrench can be represented by a unique vector $X_T \in \mathfrak{D}(\mathcal{E})$ called the moment field of T. From now on, we identify by this way any wrench T by its moment field $X_T = T$ and for example the sum $\omega_T = \omega_{X_T}$ is also called the sum of T. A wrench T is the good object to represent any system of forces acting on a rigid body. Two types of "forces" are mainly used in modeling that correspond to two types of torsors:

- a force $\vec{f}$ passing by a point $F \in \mathcal{E}$ such that $T_{\vec{f}}(F) = \vec{0}$ and $\omega_{\vec{f}} = \vec{f}$. We call such action a simple force passing by F. We deduce that $T_{\vec{f}}(M) = \vec{f} \wedge \overrightarrow{FM}$ for all $M \in \mathcal{E}$ this torsor is then denoted $F, \vec{f}$.
- a torque $\vec{\mathcal{C}}$ is a constant moment field ($\omega_{\vec{\mathcal{C}}} = \vec{0}$) with value $\vec{\mathcal{C}}$. We deduce that $T_{\vec{\mathcal{C}}}(M) = \vec{\mathcal{C}}$ for all $M \in \mathcal{E}$.

Be aware that a "force" is a whole mystery and that nobody knows what it is. We only need to have a well mathematical object to represent it. If the material system is a rigid body, the mathematical representation of any "forces system" is done by a wrench. If the material system is a particle, the SAME forces system is represented by a simple vector in E. Why?
If ϕ denotes any physical action on a body B, T_ϕ is the wrench representing the action ϕ on the body B. If there are several distinct physical actions $\phi_1, \ldots, \phi_n$ are acting on B, the wrench of the total action $\phi = \{\phi_1, \ldots, \phi_n\}$ is the sum of the different wrenches:

$$T_\phi = T_{\phi_1} + \ldots + T_{\phi_n}$$

2 Rigid body kinematics

2.1 Rigid body configuration set

Let be now B a (rigid) body. Although it is not necessary, B is viewed as a collection or assembly of particles p: $B = \{p \in B\}$. A configuration $\mathbf{s}$ of B is a placing or a locating of each particle p of B in $\mathcal{E}$. It means that $\mathbf{s}$ is viewed as a map $\mathbf{s} : B \to \mathcal{E}, p \mapsto \mathbf{s}(p) = M_p \in \mathcal{E}$ where M_p is the position of the particle p in the configuration $\mathbf{s}$. The set of configurations is noted $\mathbb{S}$. B is a rigid body if for all particles $p, q \in B$, $d(\mathbf{s}(p), \mathbf{s}(q)) = d(M_p, M_q)$ is independent on the configuration $\mathbf{s} \in \mathbb{S}$ and can be noted $d(p, q)$.

2.2 Rigid body motion

A motion of B (with respect to an observation frame $\mathcal{F} = (O; \mathcal{B})$) is a map $t \mapsto \mathbf{s}(t)$ ($\mathbb{R} \to \mathbb{S}$) or a map $t \mapsto (X_{M_p,1}(t), X_{M_p,2}(t), X_{M_p,3}(t))$ ($\mathbb{R} \to \mathbb{R}^3$) for all $p \in B$ where $M_p = O + \sum_{i=1}^{3} X_{M_p,i}(t)\vec{e}_i$.

Suppose now that, in a fixed configuration $\mathbf{r}$ (usually called a reference configuration that may also be

thought as the configuration at $t = 0$ when a motion of a body is considered), a basis is rigidly attached to the body so that at each time t the basis in the configuration $\mathbf{s}(t)$ is $\mathcal{B}_{\mathbf{s}(t)} = \mathcal{B}(t)$. Then there is a unique rotation $u_{\mathbf{s}(t)} = u(t) \in \mathcal{SO}(E)$ such that $\mathcal{B}_{\mathbf{s}(t)} = u(t)(\mathcal{B})$. If any particle or any point rigidly linked with the body denoted by C is chosen in the reference configuration,we denote by $C_{\mathbf{s}(t)} = C(t)$ the position of the same particle in the configuration $\mathbf{s}(t)$. In practice, the point C is often the inertia center of the body that will be introduced hereafter. The motion of the body B is then brought back to the motions $t \mapsto C(t) \in \mathcal{E}$ (or $t \mapsto \overrightarrow{OC}(t) \in E$ of C which describes the "position" of B by giving the position of a chosen point of B and $t \mapsto u(t) \in \mathcal{L}(E)$ which describes the attitude of B by giving the rotation passing from a given orthonormal basis associated with the observation frame (here $\mathcal{B}$ to an orthonormal basis rigidly linked with B (here $\mathcal{B}_{\mathbf{s}(t)}$).
The frame $\mathcal{F}_b = (C(t); \mathcal{B}_{\mathbf{s}(t)})$ which is moving with the body is called the body frame whereas the observation frame $\mathcal{F} = \mathcal{F}_s$ is also called the spatial frame.
Be aware that the coordinates of these geometric objects may be calculated in any basis of E for example $\mathcal{B}$ or $\mathcal{B}_{\mathbf{s}(t)}$ to get scalar or matricial unknowns. This is a real and significant difficulty without any good solution for the modeling of dynamics. Indeed, the necessary derivatives for building the scalar equation could suggest to use preferentially $\mathcal{B}$ because differentiate vectors is then reduced to differentiate their coordinates. However, kinetics of a body as we will see hereafter, is preferentially described in $\mathcal{B}_{\mathbf{s}(t)}$. Moreover, note that external actions like gravity are linked to the space and are easier described in $\mathcal{B}$ although other external actions like control or aerodynamic actions are linked to the body and are easier described in $\mathcal{B}_{\mathbf{s}(t)}$.

2.3 Rigid body velocity

Let $\mathcal{F} = (O; \mathcal{B})$ be an observation frame, let $t \mapsto \mathbf{s}(t)$ $(\mathbb{R} \to \mathbb{S})$ be a motion of B For all particle $p \in B$, the vector $\vec{v}(p, t, \mathcal{F}) = \left.\frac{d\overrightarrow{OM_p(t)}}{dt}\right)_{\mathcal{B}} \in E$ is called the velocity of the particle p at time t, with respect to the observation frame $\mathcal{F}$. That defines a vector field $\mathbb{V} = \mathbb{V}(\mathbf{s}(t)) = \mathbb{V}(t)$ on the set $D_{\mathbf{s}(t)} = \{M_p(t) = \mathbf{s}(t)(p) \mid p \in B\} \subset \mathcal{E}$ of the positions of the particles of B in configuration $\mathbf{s}(t)$ by putting $\mathbb{V}(M_p) = \vec{v}(, t, \mathcal{F})$ for all $p \in B$.
There is a close link between the dynamics of a rigid body and the following key feature concerning the field $\mathbb{V}$:

Theorem 2 *There is at each t a vector $\omega_{\mathbb{V}} = \omega_{\mathbb{V}}(t) \in E$ called the sum of $\mathbb{V}$ or the instantaneous rotation velocity of B such that for all $p, q \in B$:*

$$\mathbb{V}(M_p) = \mathbb{V}(M_q) + \omega_{\mathbb{V}} \wedge \overrightarrow{M_q M_p} \tag{10}$$

By extending this formula to the whole space $\mathcal{E}$, we deduce that $\mathbb{V} \in \mathfrak{D}(\mathcal{E})$. $\mathbb{V}$ is also called the (eulerian) velocity (vector) field of B. With the previous notations, we have at each time t:

$$\omega_{\mathbb{V}} = \omega_{\mathbb{V}(t)} = \vec{\omega}_{\mathcal{B}_{\mathbf{s}(t)}/\mathcal{B}}$$

2.4 Rigid body acceleration

In the same conditions as above, for all particle $p \in B$, the vector $\vec{\gamma}(p,t,\mathcal{F}) = \left.\frac{d\vec{v}(p,t,\mathcal{F})}{dt}\right)_{\mathcal{B}} \in E$ is called the acceleration of the particle p at time t, with respect to the observation frame $\mathcal{F}$. That allows to define an acceleration vector field $\mathbf{\Gamma}$:

$$\begin{aligned} \mathbf{\Gamma} : D_{\mathbf{s}(t)} \subset \mathcal{E} &\rightarrow \mathbb{E} \\ M = \mathbf{s}(t)(p) = M_p(t) &\mapsto \mathbf{\Gamma}(M) = \mathbf{\Gamma}(M_p(t)) = \vec{\gamma}(p,t,\mathcal{F}) \end{aligned} \tag{11}$$

and the field $\mathbf{\Gamma}$ has no property (as vector field).

3 Rigid body kinetics

Among all types of forces acting on a body, inertial forces are deeply mysterious. The origin of these forces is not really understood and the inertia principle, which is the first statement about inertia forces, is nothing else than an axiom expressing experimental results (see Galileo). Inertia forces are forces which counteract or oppose the (instantaneous) motion of any body. The existence of such forces is obvious for everybody but a distinctive difficulty lies in its mathematical expression. Indeed, it depends on the observation frame and this expression will be calculable only in inertial frames. These frames will be defined in the following section. When one speaks about motion, it means velocity and acceleration fields investigated in the previous section, which ,we should point out, never specified the type of observation frames: the whole previous section is valid for any frame. Every body knows that the inertia force acting on a particle p (when the motion is observed with respect to an inertia frame $\mathcal{F}$) is $-m\vec{\gamma}(p,t,\mathcal{F})$. The scalar m is called the mass of p and must be thought as the "good parameter" in order to find the good movement equations meaning the equations whose solutions are the observed trajectories. First it may not be confused with the gravitational mass and sometimes one uses the word inertial mass not to do confusing. Its relationship with the gravitational mass is however very deep and is clearly highlighted only by relativity theory. As far as we are concerned, both types of masses may be and will be confused. The present section focuses on how extending the mass concept only valid for a particle to a whole rigid body which is called the kinetics of the body. We also investigate how extending the momentum $m\vec{v}(p,t,\mathcal{F})$ and the kinetic energy $\frac{1}{2}m\vec{v}^2(p,t,\mathcal{F})$ of a particle to a whole rigid body.

3.1 Mass axiom, total mass, center of inertia

Axiom 1 *For each body B and each configuration $\mathbf{s} \in \mathbb{S}$ of B, there is a map called (volume) density $\rho_{\mathbf{s}} : D_{\mathbf{s}} \to \mathbb{R}_+$ giving the mass distribution of B in configuration $\mathbf{s}$. If the spatial position of $p \in B$ is $\mathbf{s}(p) = M_p \in D_{\mathbf{s}}$ then the mass of an elementary volume $dv(M_p)$ at M_p is $\rho_{\mathbf{s}}(M_p)dv(M_p)$. This map is supposed $\mathbb{D}(\mathcal{E})$- invariant: $\rho_{\mathbf{s}}(A(\mathbf{s}(p))) = \rho_{\mathbf{s}}(\mathbf{s}(p)) = \rho_{\mathbf{s}}(M_p)$ for all $A \in \mathbb{D}(\mathcal{E})$ or in other words*

$$\rho_{A\bullet\mathbf{s}} = \rho_{\mathbf{s}} \quad \forall A \in \mathbb{D}(\mathcal{E}) \;\; \forall \mathbf{s} \in \mathbb{S}$$

Because of the invariance property, we put $\rho(p) = \rho_{\mathbf{s}}(\mathbf{s}(p))$ which is not depending on $\mathbf{s} \in \mathbb{S}$. It should be noted that $dv(M_p)$ is the Lebesgue measure at M_p in $\mathcal{E} \approx \mathbb{R}^3$. Sometimes used models are linear or planar so that $dv(M_p)$ then becomes the Lebesgue measure at M_p in $\mathbb{R}$ or $\mathbb{R}^2$. It is then necessary to introduce the following definitions:

Definition 2 • *the* **total mass** m *of* B *is*

$$m = \int_{p \in B} \rho(p) dv(M_p) \tag{12}$$

- *for all configuration* $\mathbf{s}$ *the* **center of inertia or center of mass** $G_{\mathbf{s}} \in \mathcal{E}$ *is the unique point of* $\mathcal{E}$ *defined by*

$$\int_{p \in B} \overrightarrow{G_{\mathbf{s}} M_p} \rho(p) dv(M_p) = \vec{0} \tag{13}$$

Be however careful that the point $G_{\mathbf{s}} \in \mathcal{E}$ does not necessary correspond to a material particle of B (Think for example to a hoop whose the center of inertia is the geometric center of the corresponding circle where there is no matter). The center of inertia can be however considered as an element of B because $d(\mathbf{s}(p), G_{\mathbf{s}})$ is not depending on $\mathbf{s}$ and depends only on $p \in B$. Such a point may be called a point rigidly linked to the body B or, more briefly, a rigid linked point (r.l.p) (or sometimes also a point moving with the body). Then, G will refer, somewhat imprecisely, to a point of B so that $G_{\mathbf{s}}$ could be the position of G in the configuration $\mathbf{s}$. In other words, each configuration $\mathbf{s}$ thought as placement of "particles" of B is naturally and implicitly extended to rigid linked points. For example, $G_{\mathbf{s}} = \mathbf{s}(G)$. From (13) we deduce that, for all $\mathbf{s} \in \mathbb{S}$:

$$m\overrightarrow{PG_{\mathbf{s}}} = \int_{p \in B} \overrightarrow{PM_p} \rho(p) dv(M_p) \quad \forall P \in \mathcal{E} \tag{14}$$

3.2 Inertia operator, inertia matrix

Definition 3

Let now F *be an element of* B *(or a r.l.p)). For all configuration* $\mathbf{s}$ *the* **inertia operator or inertia tensor with respect to** F $\mathbb{I}_F(\mathbf{s})$ *is the element of* $\mathcal{L}(\mathbb{E})$ *defined by:*

$$\mathbb{I}_F(\mathbf{s})(\vec{x}) = \int_{p \in B} \overrightarrow{F_{\mathbf{s}} M_p} \wedge (\vec{x} \wedge \overrightarrow{F_{\mathbf{s}} M_p}) \rho(p) dv(p) \quad \forall \vec{x} \in \mathbb{E} \tag{15}$$

For $F = G$, $\mathbb{I}_G$ is the central inertia operator and often simply noted $\mathbb{I}$. The following proposition holds:

Proposition 3 *For all configuration* $\mathbf{s} \in \mathbb{S}$, $\mathbb{I}_F(\mathbf{s})$*is a symmetric operator of the euclidean space* $(\mathbb{E}, (. \mid .))$ *and for all* $\vec{x}, \vec{y} \in \mathbb{E}$*:*

$$(\vec{x} \mid \mathbb{I}_F(\mathbf{s})(\vec{y})) = (\mathbb{I}_F(\mathbf{s})(\vec{x}) \mid y) = \int_{p \in B} (\vec{x} \wedge \overrightarrow{F_{\mathbf{s}} M_p} \mid \vec{y} \wedge \overrightarrow{F_{\mathbf{s}} M_p}) \rho(p) dv(p) \tag{16}$$

It follows that the matrix of $\mathbb{I}_F(\mathbf{s})$ in any **orthonormal** basis of E is itself symmetric. m, G and for example $\mathbb{I}$ are necessary and enough to completely describe the kinetics of B meaning that only $1 + 3 + 6 = 10$ scalars are enough to to completely describe the kinetics of B.

Let now $\mathcal{B} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be an orthonormal basis and $\mathcal{F} = (F_{\mathbf{s}}; \mathcal{B})$ the associated coordinate system with $F_{\mathbf{s}}$ as origin. If $p \in B$, $(x_{p,\mathbf{s},1}, x_{p,\mathbf{s}2}, x_{p,\mathbf{s},3}) = (x_{p,1}, x_{p,2}, x_{p,3})$ (forgetting intentionally the dependency on $\mathbf{s}$) are the coordinates of $\mathbf{s}(p) = M_p$ in $(F_{\mathbf{s}}; \mathcal{B})$ so that $\overrightarrow{F_{\mathbf{s}}M_p} = \sum_{i=1}^{3} x_{p,i}\vec{e}_i$. Straightforward calculations give:

Proposition 4 *The matrix of $\mathbf{I}_{\mathcal{B},F}(\mathbf{s})$ of $\mathbb{I}_F(\mathbf{s})$ in $\mathcal{B}$ is:*

$$\mathbf{I}_{\mathcal{B},F}(\mathbf{s}) = \begin{pmatrix} \int_{p\in B}(x_{p,2}^2 + x_{p,3}^2)\rho(p)dv(p) & -\int_{p\in B} x_{p,1}x_{p,2}\rho(p)dv(p) & -\int_{p\in B} x_{p,1}x_{p,3}\rho(p)dv(p \\ -\int_{p\in B} x_{p,2}x_{p,1}\rho(p)dv(p & \int_{p\in B}(x_{p,1}^2 + x_{p,3}^2)\rho(p)dv(p) & -\int_{p\in B} x_{p,2}x_{p,3}\rho(p)dv(p \\ -\int_{p\in B} x_{p,3}x_{p,1}\rho(p)dv(p & -\int_{p\in B} x_{p,3}x_{p,2}\rho(p)dv(p & \int_{p\in B}(x_{p,1}^2 + x_{p,2}^2)\rho(p)dv(p) \end{pmatrix} \quad (17)$$

Be careful that this matrix is depending on $B, F, \mathcal{B}$ but also on $\mathbf{s}$. If $\mathcal{B}$ is not attached to $\mathbf{s}$ or in other words rigidly linked to B, this matrix becomes a time function during the motion $t \mapsto \mathbf{s}(t)$ of B. This seemingly trivial remark is in fact the explanation of the not removable complexity of scalar dynamic equations.
By the so-called Huygens theorem, $\mathbb{I}_F$ and $\mathbb{I}$ are in relationship by :

Proposition 5

$$\mathbb{I}_F(\mathbf{s}) = \mathbb{I}(\mathbf{s}) + m\overrightarrow{G_{\mathbf{s}}F_{\mathbf{s}}} \wedge (. \wedge \overrightarrow{G_{\mathbf{s}}F_{\mathbf{s}}}) \quad (18)$$

meaning

$$\mathbb{I}_F(\mathbf{s})(\vec{x}) = \mathbb{I}(\mathbf{s})(\vec{x}) + m\overrightarrow{G_{\mathbf{s}}F_{\mathbf{s}}} \wedge (\vec{x} \wedge \overrightarrow{G_{\mathbf{s}}F_{\mathbf{s}}}) \quad \forall \vec{x} \in E \quad (19)$$

or in a matricial point of view for all $i \neq j \neq k$

$$\mathbf{I}_{\mathcal{B},F}(\mathbf{s})_{ii} = \mathbf{I}_{\mathcal{B}}(\mathbf{s})_{ii} + m(x_{j,F_{\mathbf{s}}}^2 + x_{k,F_{\mathbf{s}}}^2) \quad (20)$$

$$\mathbf{I}_{\mathcal{B},F}(\mathbf{s})_{ij} = \mathbf{I}_{\mathcal{B}}(\mathbf{s})_{ij} - m x_{i,F_{\mathbf{s}}} x_{j,F_{\mathbf{s}}} \quad (21)$$

with $(x_{1,F_{\mathbf{s}}}, x_{2,F_{\mathbf{s}}}, x_{3,F_{\mathbf{s}}})$ coordinates of $\overrightarrow{G_{\mathbf{s}}F_{\mathbf{s}}}$ in **B**.

3.3 Kinetic wrench or kinetic moment field

The goal of this section is to combine the elements of kinetics and kinematics in order to be closer to our announced goal concerning the forces of inertia. We adopt the same notations as above concerning motion and kinetics. For the moment, the observation frame $\mathcal{F} = (O; \mathcal{B})$ is propertyless. Let $t \mapsto \mathbf{s}(t)$ the motion of the body B observed with respect to $\mathcal{F}$.

Definition 4 *The kinetic wrench or kinetic moment field $\mathbb{H} = \mathbb{H}_{\mathbf{s}(t)}$ of B at the configuration $\mathbf{s}(t)$ (meaning at time t) is the vector field on $\mathcal{E}$ defined by:*

$$\mathbb{H}_{\mathbf{s}(t)}(M) = \mathbb{H}(M) = \int_{p\in B} \overrightarrow{MM_p(t)} \wedge \vec{v}(p,t,\mathcal{F})\rho(p)dv(p) = \int_{p\in S} \overrightarrow{MM_p(t)} \wedge \mathbb{V}(M_p(t))\rho(p)dv(p) \quad \forall M \in \mathcal{E} \quad (22)$$

where, to be remembered, $M_p = \mathbf{s}(t)(p) = M_p(t)$ for all $p \in B$. The sum of the wrench $\mathbb{H} = \mathbb{H}_{\mathbf{s}(t)}$ is $\omega_{\mathbb{H}} = m\mathbb{V}(G_{\mathbf{s}(t)})$ and is called **the momentum** *of B. For all $M \in \mathcal{E}$, $\mathbb{H}_{\mathbf{s}(t)}(M)$ is called* **the kinetic or angular momentum of B in the configuration $\mathbf{s}(t)$ at the point M**.

Remark that M is any point of $\mathcal{E}$ and not necessarily linked to B. However:

Proposition 6 *for any F element of B (or a r.l.p)), and for any t*

$$\mathbb{H}(F_{\mathbf{s}(t)}) = \mathbb{I}_F(\mathbf{s}(t))(\omega_{\mathbb{V}}) + m\overrightarrow{F_{\mathbf{s}(t)}G_{\mathbf{s}(t)}} \wedge \mathbb{V}(F_{\mathbf{s}(t)}) \tag{23}$$

□

$$\begin{aligned}
\mathbb{H}(F_{\mathbf{s}(t)}) &= \int_{p\in B} \overrightarrow{F_{\mathbf{s}(t)}M_p(t)} \wedge \mathbb{V}(M_p(t))\rho(p)dv(p) \\
&= \int_{p\in B} \overrightarrow{F_{\mathbf{s}(t)}M_p(t)} \wedge (\mathbb{V}(F_{\mathbf{s}(t)}) + \omega_{\mathbb{V}} \wedge \overrightarrow{F_{\mathbf{s}(t)}M_p(t)})\rho(p)dv(p) \\
&= \int_{p\in B} \overrightarrow{F_{\mathbf{s}(t)}M_p(t)} \wedge \mathbb{V}(F_{\mathbf{s}(t)})\rho(p)dv(p) + \int_{p\in B} \overrightarrow{F_{\mathbf{s}(t)}M_p(t)} \wedge (\omega_{\mathbb{V}} \wedge \overrightarrow{F_{\mathbf{s}(t)}M_p}(t))\rho(p)dv(p) \\
&= m\overrightarrow{F_{\mathbf{s}(t)}G_{\mathbf{s}(t)}} \wedge \mathbb{V}(F_{\mathbf{s}(t)}) + \mathbb{I}_F(\mathbf{s}(t))(\omega_{\mathbb{V}})
\end{aligned}$$

■

Two particular cases have to be kept in mind:

- If F is motionless in $\mathcal{F}$ then
$$\mathbb{H}(F_{\mathbf{s}(t)}) = \mathbb{I}_F(\mathbf{s}(t))(\omega_{\mathbb{V}}) \tag{24}$$
- If $F \equiv G$ is the center of inertia of B then
$$\mathbb{H}(G_{\mathbf{s}(t)}) = \mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}}) \tag{25}$$

3.4 Kinetic energy

As above, with the same notations,

Definition 5 *The knetic energy $C_{\mathbf{s}(t)}$ of B in the configuration $\mathbf{s}(t)$ is the scalar:*

$$C_{\mathbf{s}(t)} = \frac{1}{2}\int_{p\in B} (\vec{v}(p,t,\mathcal{R}) \mid \vec{v}(p,t,\mathcal{F}))\rho(p)dv(p) \tag{26}$$

Following expressions of $C_{\mathbf{s}(t)}$ are useful

Proposition 7

$$C_{\mathbf{s}(t)} = \frac{1}{2}[\mathbb{H} \mid \mathbb{V}] = \frac{1}{2}m\mathbb{V}(G_{\mathbf{s}(t)})^2 + \frac{1}{2}(\mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}}) \mid \omega_{\mathbb{V}}) \tag{27}$$

□ The proof is let to the reader ■

4 Rigid body dynamics

4.1 Dynamic wrench (or torsor) or dynamic moment field

By an analogous way as for the kinetic wrench, we are led to put the

Definition 6 *The dynamic wrench or dynamic moment field* $\mathbb{K} = \mathbb{K}_{\mathbf{s}(t)}$ *of* B *at the configuration* $\mathbf{s}(t)$ *(meaning at time* t*) is the vector field on* $\mathcal{E}$ *defined by:*

$$\mathbb{K}_{\mathbf{s}(t)}(M) = \mathbb{K}(M) = \int_{p\in B} \overrightarrow{MM_p(t)} \wedge \vec{\gamma}(p,t,\mathcal{F})\rho(p)dv(p) = \int_{p\in S} \overrightarrow{MM_p(t)} \wedge \mathbf{\Gamma}(M_p(t))\rho(p)dv(p) \quad \forall M \in \mathcal{E} \tag{28}$$

where, to be remembered, $M_p = \mathbf{s}(t)(p) = M_p(t)$ *for all* $p \in B$.

The sum of the whrench $\mathbb{K} = \mathbb{K}_{\mathbf{s}(t)} = \mathbb{K}(t)$ is $\omega_{\mathbb{K}} = m\mathbf{\Gamma}(G_{\mathbf{s}(t)})$.
For all $M \in \mathcal{E}$, $\mathbb{K}_{\mathbf{s}(t)}(M)$ is called the dynamic moment of B in the configuration $\mathbf{s}(t)$ at the point M. Remark again that M is any point of $\mathcal{E}$ and not necessarily linked to B. Remark mainly that $\mathbb{K}$ is not the derivative of $\mathbb{H}$ even if the acceleration is the derivative of the velocity. The relationship between $\mathbb{H}$ and $\mathbb{K}$ is given by:

Proposition 8 *For each* $M \in \mathcal{E}$ *and at each time* t*:*

$$\mathbb{K}(M) = \left.\frac{d\mathbb{H}(M)}{dt}\right)_{\mathcal{B}} + m\vec{v}(M,t,\mathcal{F}) \wedge \mathbb{V}(G_{\mathbf{s}(t)}) \tag{29}$$

where $\vec{v}(M,t,\mathcal{F}) = \left.\frac{d\overrightarrow{OM}}{dt}\right)_{\mathcal{B}}$

□ Applying the derivative rules ($\wedge$ is differentiated as a product because it is bilinear):

$$\begin{aligned}
\left.\frac{d\mathbb{H}(M)}{dt}\right)_{\mathcal{F}} &= \left.\frac{d\int_{p\in B}\overrightarrow{MM_p(t)} \wedge \vec{v}(p,t,\mathcal{F})\rho(p)dv(p)}{dt}\right)_{\mathcal{F}} \\
&= \int_{p\in B} \left.\frac{d\overrightarrow{MM_p(t)}}{dt}\right)_{\mathcal{F}} \wedge \vec{v}(p,t,\mathcal{F})\rho(p)dv(p) + \int_{p\in B} \overrightarrow{MM_p(t)} \wedge \left.\frac{d\vec{v}(p,t,\mathcal{F})}{dt}\right)_{\mathcal{F}} \rho(p)dv(p) \\
&= \int_{p\in B} (\mathbb{V}(M_p(t)) - \vec{v}(M,t,\mathcal{F})) \wedge \vec{v}(p,t,\mathcal{F})\rho(p)dv(p) + \int_{p\in B} \overrightarrow{MM_p(t)} \wedge \vec{\gamma}(p,t,\mathcal{F})\rho(p)dv(p) \\
&= -\int_{p\in B} (\vec{v}(M,t,\mathcal{F})) \wedge \vec{v}(p,t,\mathcal{F})\rho(p)dv(p) + \mathbb{K}(M) \\
&= -m\vec{v}(M,t,\mathcal{F})) \wedge \mathbb{V}(G_{\mathbf{s}(t)}) + \mathbb{K}(M)
\end{aligned}$$

■
Two important cases have to be remembered:

- If F is motionless with respect to $\mathcal{F}$ then

$$\mathbb{K}(F_{\mathbf{s}(t)}) = \left.\frac{d\mathbb{H}(F_{\mathbf{s}(t)})}{dt}\right)_{\mathcal{B}} = \left.\frac{d(\mathbb{I}_F(\mathbf{s}(t))(\omega_{\mathbb{V}}))}{dt}\right)_{\mathcal{B}} \tag{30}$$

- If $F \equiv G$ is the center of inertia, then

$$\mathbb{K}(G_{\mathbf{s}(t)}) = \left.\frac{d\mathbb{H}(G_{\mathbf{s}(t)})}{dt}\right)_{\mathcal{B}} = \left.\frac{d(\mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}}))}{dt}\right)_{\mathcal{B}} \tag{31}$$

The last relation may be specified by the following:

Theorem 9

$$\left.\frac{d(\mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}}))}{dt}\right)_{\mathcal{B}} = \mathbb{I}(\mathbf{s}(t))(\left.\frac{d(\omega_{\mathbb{V}})}{dt}\right)_{\mathcal{B}}) + \omega_{\mathbb{V}} \wedge \mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}}) \tag{32}$$

□ According to the usual derivative rules:

$$\left.\frac{d\mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}})}{dt}\right)_{\mathcal{R}} = \mathbb{I}(\mathbf{s}(t))(\left.\frac{d(\omega_{\mathbb{V}})}{dt}\right)_{\mathcal{B}}) + \left.\frac{d\mathbb{I}(\mathbf{s}(t))}{dt}\right)_{\mathcal{B}}(\omega_{\mathbb{V}}) \tag{33}$$

Operator $\mathbb{I}(\mathbf{s}(t)) = \mathbb{I}(t)$ must then be differentiated over time and the result is given by the following

Lemma 1 *For all t and $\vec{x} \in \mathbb{E}$:*

$$\frac{d\mathbb{I}(\mathbf{s}(t))}{dt}(\vec{x}) = \omega_{\mathbb{V}} \wedge \mathbb{I}(\mathbf{s}(t))(\vec{x}) + \mathbb{I}(\mathbf{s}(t))(\vec{x} \wedge \omega_{\mathbb{V}}) \tag{34}$$

□ Forgetting intentionally $\mathcal{F}, \mathcal{B}, \mathbf{s}(t), t, \ldots$ by sake of simplicity:

$$\begin{aligned}
\frac{d\mathbb{I}(\mathbf{s}(t))}{dt}(\vec{x}) &= \frac{d}{dt}(\int_{p\in B} \overrightarrow{G_{\mathbf{s}}M_p^{\mathbf{s}}} \wedge (\vec{x} \wedge \overrightarrow{G_{\mathbf{s}}M_p})\rho(p)dv(p)) \\
&= \int_{p\in B} \frac{d\overrightarrow{G_{\mathbf{s}}M_p}}{dt} \wedge (\vec{x} \wedge \overrightarrow{G_{\mathbf{s}}M_p})\rho(p)dv(p) + \int_{p\in B} \overrightarrow{G_{\mathbf{s}}M_p} \wedge (\vec{x} \wedge \frac{d\overrightarrow{G_{\mathbf{s}}M_p}}{dt})\rho(p)dv(p) \\
&= \int_{p\in B} \frac{d\overrightarrow{G_{\mathbf{s}}M_p}}{dt} \wedge (\vec{x} \wedge \overrightarrow{G_{\mathbf{s}}M_p})\rho(p)dv(p) + \int_{p\in B} \overrightarrow{G_{\mathbf{s}}M_p} \wedge (\vec{x} \wedge \frac{d\overrightarrow{G_{\mathbf{s}}M_p}}{dt})\rho(p)dv(p) \\
&= \int_{p\in B} (\omega_{\mathbb{V}} \wedge \overrightarrow{G_{\mathbf{s}}M_p}) \wedge (\vec{x} \wedge \overrightarrow{G_{\mathbf{s}}M_p})\rho(p)dv(p) + \int_{p\in B} \overrightarrow{G_{\mathbf{s}}M_p} \wedge (\vec{x} \wedge (\omega_{\mathbb{V}} \wedge \overrightarrow{G_{\mathbf{s}}M_p}))\rho(p)dv(p) \\
&= \int_{p\in B} (\omega_{\mathbb{V}} \wedge \overrightarrow{G_{\mathbf{s}}M_p}) \wedge (\vec{x} \wedge \overrightarrow{G_{\mathbf{s}}M_p}) + \overrightarrow{G_{\mathbf{s}}M_p} \wedge (\vec{x} \wedge (\omega_{\mathbb{V}} \wedge \overrightarrow{G_{\mathbf{s}}M_p}))\rho(p)dv(p) \\
&= \int_{p\in B} (\omega_{\mathbb{V}} \wedge \overrightarrow{G_{\mathbf{s}}M_p}) \wedge (\vec{x} \wedge \overrightarrow{G_{\mathbf{s}}M_p}) - \overrightarrow{G_{\mathbf{s}}M_p} \wedge (\omega_{\mathbb{V}} \wedge (\overrightarrow{G_{\mathbf{s}}M_p} \wedge \vec{x}) + \overrightarrow{G^{\mathbf{s}}M_p^{\mathbf{s}}} \wedge (\vec{x} \wedge \omega_{\mathbb{V}}))\rho(p)dv(p) \\
&= \int_{p\in B} (\omega_{\mathbb{V}} \wedge \overrightarrow{G_{\mathbf{s}}M_p}) \wedge (\vec{x} \wedge \overrightarrow{G_{\mathbf{s}}M_p}) - \overrightarrow{G_{\mathbf{s}}M_p} \wedge (\omega_{\mathbb{V}} \wedge (\overrightarrow{G_{\mathbf{s}}M_p} \wedge \vec{x}))\rho(p)dv(p) \\
&\quad - \int_{p\in B} \overrightarrow{G_{\mathbf{s}}M_p} \wedge (\overrightarrow{G_{\mathbf{s}}M_p} \wedge (\vec{x} \wedge \omega_{\mathbb{V}}))\rho(p)dv(p) \\
&= \int_{p\in B} \omega_{\mathbb{V}} \wedge (\overrightarrow{G_{\mathbf{s}}M_p} \wedge (\vec{x} \wedge \overrightarrow{G_{\mathbf{s}}M_p}))\rho(p)dv(p) + \int_{p\in B} \overrightarrow{G_{\mathbf{s}}M_p} \wedge ((\vec{x} \wedge \omega_{\mathbb{V}}) \wedge \overrightarrow{G_{\mathbf{s}}M_p})\rho(p)dv(p) \\
&= \omega_{\mathbb{V}} \wedge \int_{p\in B} \overrightarrow{G_{\mathbf{s}}M_p} \wedge (\vec{x} \wedge \overrightarrow{G_{\mathbf{s}}M_p})\rho(p)dv(p) + \int_{p\in B} \overrightarrow{G_{\mathbf{s}}M_p} \wedge ((\vec{x} \wedge \omega_{\mathbb{V}}) \wedge \overrightarrow{G_{\mathbf{s}}M_p})\rho(p)dv(p) \\
&= \omega_{\mathbb{V}} \wedge \mathbb{I}(\mathbf{s}(t))(\vec{x}) + \mathbb{I}(\mathbf{s}(t))(\vec{x} \wedge \omega_{\mathbb{V}})
\end{aligned}$$

by applying as many times as necessary the Jacobi's identity (1). ∎

When the previous relation is applied to $\vec{x} = \omega_{\mathbb{V}}$, it follows that $\dfrac{d\mathbb{I}(\mathbf{s}(t))}{dt}(\omega_{\mathbb{V}}) = \omega_{\mathbb{V}} \wedge \mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}})$ which leads exactly to (32).

4.2 System of forces acting on a rigid body

In this paragraph, we give some precisions about what means a "force" acting on a body which has been already introduced in the paragraph on duality. Because of the rigidity of the body B, any physical action ϕ or force acting on B in the configuration $\mathbf{s} \in \mathbb{S}$ may be represented or described by a torsor or wrench $T_\phi = T_{\phi,\mathbf{s}}$. If several actions $\phi_1, \ldots, \phi_n$ (setting a so called system of forces) are acting on B, then the system of forces $\phi = \phi_1 \cup \ldots \cup \phi_n$ may be itself described by the unique wrench $T_{\phi,\mathbf{s}} = T_{\phi_1,\mathbf{s}} + \ldots + T_{\phi_n,\mathbf{s}}$. These actions are external because we consider that the sources of these actions are coming from domains $S \subset \mathcal{E}$ such that $S \cap D_\mathbf{s} = \emptyset$. The equivalent torsor of all these external action is denoted $T_{ext,\mathbf{s}}$. Sometimes it can be interesting to distinguish two kind of external actions on a body: those linked to the body B that we can call body forces noted $T^b_{ext,\mathbf{s}}$ and those linked to the space $\mathcal{E}$ that we will call space forces and noted $T^s_{ext,\mathbf{s}}$. Moreover, these torsors are functions of several quantities. Among these quantities, some are data of the problem while other are unknowns. Here, unknowns mean motion or kinematic unknowns or with notations used above $\mathbf{s}, \mathbb{V}, \mathbf{\Gamma}$ or $\mathbf{r}, A, \mathbb{V}, \mathbf{\Gamma}$. It is commonly assumed that only $\mathbf{s}, \mathbb{V}$ or $\mathbf{r}, A, \mathbb{V}$ are explicitly variables. If only $\mathbf{s}$ or $\mathbf{r}, A$ appear, one speaks about positional forces. Gravity or buoyancy are for example position actions on a body. On the contrary, wind action on a body are depending on $\mathbb{V}$ and even on $\mathbf{\Gamma}$ if fluid-structure interactions are considered which leads to added mass effects. Theses points will be specified in the last section as well as for control actions. Generally, it is considered that $T_{ext,\mathbf{s}}$ introduce no other unknown than those of motion. They are also called given-actions. If the body is subjected to actions that constrains the motion of the body (in a kinematical view point), the corresponding action is called a link-action. For flying objects, there is no external action that could kinematically constrain the system: for such system, except if the modeling leads to such constraints, the external actions are given actions.

Two other "sources" of forces may be considered. The first ones are internal forces. As in continuum mechanics, these actions are thought by cutting virtually B into two any parts $D_{1,\mathbf{s}}$ and $D_{2,\mathbf{s}}$ in the configuration $\mathbf{s}$ and considering the mutual interactions $T_{12,\mathbf{s}} = -T_{12,\mathbf{s}}$ of the part 2 on the part 1. A principle of mechanics claims that if B has a rigid motion, the **sum** of all these actions is nil: they cannot appear in the dynamic equations of a rigid body even if it should be false to think that there is no internal actions!!! For a deformable system, the internal actions are generally contribute to supplementary unknowns in dynamic equations. These actions obviously should be classified in link-actions because they constrain the body to remain rigid!! The second "source" is this of inertia forces. As already mentioned, the physical source of these forces is not clear but as for any forces systems acting on a rigid body these forces are equivalent to a torsor or a wrench which will be, from now on, denoted T_{in} or $T_{in,\mathbf{s}}$.

4.3 Dynamic Principle of a rigid body

At the moment, we do not specify the observation frame $\mathcal{F}$. The dynamic principle reads:

Axiom 2 *At each time t,*

$$T_{ext,\mathbf{s}(t)} + T_{in,\mathbf{s}(t)} = 0 \tag{35}$$

This is an equation in $\mathfrak{D}(\mathcal{E})$ which is equivalent to 6 scalar equations. Because the expression of $T_{in,\mathbf{s}}$ as function of motion unknowns is itself unknown if $\mathcal{F}$ is any, this formulation of dynamic equations is not very useful. This is the reason why a specific class of frames has been introduced. These are the so-called inertial frames which must be defined by an

Axiom 3 *There is a class of observation frames called the set of inertial frames, noted $\mathcal{I}_{\mathcal{F}}$ such that:*

- *if $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{I}_{\mathcal{F}}$ the motion of $\mathcal{F}_1$ with respect to $\mathcal{F}_2$ is an uniform rectilinear translatory motion.*
- *If $t \mapsto \mathbf{s}(t)$ is the motion of a rigid body B with respect to $\mathcal{F} \in \mathcal{I}_{\mathcal{F}}$ then $T_{in} = -\mathbb{K}$.*

Remark that only experimental observations may justify that a given frame is an inertial frame. It is actually never rigorously right but only approximately valid. A modeler has to assume that a convenient frame is an inertial frame. In some situations a frame may be supposed inertial while for other types of applications, the same frame cannot more assumed to be an inertial frame.
Suppose then now that the motion is observed with respect to $\mathcal{F} \in \mathcal{I}_{\mathcal{F}}$. We deduce that for all t,

$$\mathbb{K} = T_{ext,\mathbf{s}(t)} \tag{36}$$

(36) may be brought back to two independent equations in E by making equal the sums and the values at a convenient point. As is often the case, the center of inertia is the chosen point which leads to the following system of equations in E. Then, supposing that external actions are built by a family of n simple forces $(\vec{f_k}, F_k)_{k=1,\dots,n}$ and a family of m torques $(\vec{\mathcal{C}_j})_{j=1,\dots,m}$, for all t

$$m\mathbf{\Gamma}(G_{\mathbf{s}(t)}) = m \left.\frac{d\mathbb{V}(G_{\mathbf{s}(t)})}{dt}\right)_{\mathcal{B}} = \omega_{T_{ext,\mathbf{s}(t)}} = \sum_{k=1}^{n} \vec{f_k} \tag{37}$$

$$\left.\frac{d\mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}})}{dt}\right)_{\mathcal{B}} = T_{ext,\mathbf{s}(t)}(G_{\mathbf{s}(t)}) \qquad \text{or more accurately}$$

$$\left.\frac{d\mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}})}{dt}\right)_{\mathcal{B}} = \mathbb{I}(\mathbf{s}(t))(\left.\frac{d(\omega_{\mathbb{V}})}{dt}\right)_{\mathcal{B}}) + \omega_{\mathbb{V}} \wedge \mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}}) = \sum_{k=1}^{n} \overrightarrow{G_{\mathbf{s}(t)}F_k} \wedge \vec{f_k} + \sum_{j=1}^{m} \vec{\mathcal{C}_j} \tag{38}$$

Of course, $F_k = F_{k,\mathbf{s}(t)}$ as well for $\vec{f_k}$ and $\vec{\mathcal{C}_j}$. (37) governs the position of the center of inertia whereas (38) governs the attitude of the body although, obviously, the two equations are strongly coupled. These equations are vectorial and may be projected in any basis of E. The choice of the basis is extremely significant.
Two kinds of bases may occur: those (rigidly) linked to the observation frame and those (rigidly) linked to the body. With the first choice, derivatives are easy to calculate while the matrix of $\mathbb{I}$ in such a basis is time depending. With the second choice, derivatives are more complicated to calculate while the matrix of $\mathbb{I}$ in such a basis is constant. Although there is no perfectly "good" choice, the second one is almost always done. (37) and (38) appear as first order differential equations with vector field $\mathbb{V}$ as unknown. If the/a displacement field $(\mathbf{r}, A)$ is chosen as unknown, the relationship between $\mathbb{V}$ and A must be added. In any

event, the displacement field explicitly appears in the right-hand side of these equations must rarely or never be eliminated. Remark also that if $G_{\mathbf{r}}$ is the position of the center of inertia in the reference configuration, then $G_{\mathbf{s}(t)} = A(t)(G_{\mathbf{r}})$ so that $\mathbb{V}(G_{\mathbf{s}(t)}) = \left.\frac{d\,\overrightarrow{G_{\mathbf{r}}A(t)G_{\mathbf{r}}}}{dt}\right)_{\mathcal{B}}$. The relationship between $\mathbf{r}, A$ and $\mathbb{V}$ reads then:

$$\mathbb{V}(G_{\mathbf{s}(t)}) = \left.\frac{d\,\overrightarrow{G_{\mathbf{r}}A(t)G_{\mathbf{r}}}}{dt}\right)_{\mathcal{B}} \tag{39}$$

$$\omega_{\mathbb{V}} = \vec{\omega}_{\mathcal{B}(t)/\mathcal{B}} \tag{40}$$

where $\mathcal{B} = \mathcal{B}(t)$ is a basis rigidly linked with the body and $\mathcal{B}$ a basis linked (rigidly) with the observation frame $\mathcal{F}$.

Equations (37) and (38) are respectively the Newton equation and the Euler equation and both set up the dynamic modeling of the body. Both equations (39) and (40) set up the kinematic modeling of the body.

We now specify some elements concerning the command of flying objects.

5 Case of flying objects

5.1 Some general features about dynamic equations

Once the vector equations (37) and (38) in the hands, another modeling work begins. This mainly concerns the three following points:

- the choice of reference frame
- the choice of the projection basis
- the description of the external "natural" forces
- the description of the control forces

Concerning the first issue, this choice is depending on the concrete problem. Finally, it is necessary to fix an inertial frame. Typically for a wide class of applications, a frame linked with the earth can be considered as an inertial frame. Sometimes the rotation of the earth cannot be neglected. In this case, the inertial frame is built with the center of the earth as origin and three orthogonal directions fixed in $\mathcal{E}$. We have already discussed the second point and we concluded that a well adapted basis is anyone basis linked to the body B. For a such choice, the inertia matrix has constant coefficients as well as a great part of forces, especially for power-control forces and wind forces. Once this point admitted, several situations may occur according to the involved issue: equilibrium stabilization, trajectory tracking, uniform and constant wind, transitory conditions and so one, each of these particular conditions leading to a specific treatment of the general equations. Here, for the sake of simplicity, we distinguish two kind of external actions on B: the "natural" forces and the control forces. The first ones includes wind forces , gravity, eventually buoyancy whereas the second ones mainly includes propulsion and driving system forces. We successively consider theses two types.

5.2 Natural forces

The only modeling we will discuss about concerns the wind action because we may suppose that other natural external actions are clear to model. The actions are mainly considered as functions of the position $\mathbf{s}$ and the velocity $\mathbb{V}$ of the body.
The gravity action is described by a torsor T_g. It is a simple force passing through G and only depending on the position of G (center of inertia and center of gravity are identified) meaning that $T_{g,\mathbf{s}} = T_{g,\mathbf{s}}(\mathbf{s})$ so that:

$$T_{g,\mathbf{s}}(\mathbf{s}) = \left\{ \begin{array}{rcl} \omega_{T_{g,\mathbf{s}}(\mathbf{s})} & = & m\vec{g} \\ T_{g,\mathbf{s}}(\mathbf{s})(G_\mathbf{s}) & = & \vec{0} \end{array} \right\}_{G_\mathbf{s}}$$

where $\vec{g}$ is the gravity acceleration directed downward vertically.
The buoyancy, according to Archimed's law, affects every body in relation to the amount of air it displaces. It follows that buoyancy is described by a torsor T_b. It is a simple force passing through C_b and only depending on the position $C_{b,\mathbf{s}}$ of C_b in the configuration $\mathbf{s}$ (geometric center of $D_\mathbf{s}$) meaning that $T_{b,\mathbf{s}} = T_{b,\mathbf{s}}(\mathbf{s})$ so that:

$$T_{b,\mathbf{s}}(\mathbf{s}) = \left\{ \begin{array}{rcl} \omega_{T_{b,\mathbf{s}}(\mathbf{s})} & = & -m_f\vec{g} \\ T_{b,\mathbf{s}}(\mathbf{s})(C_{b,\mathbf{s}}) & = & \vec{0} \end{array} \right\}_{C_{b,\mathbf{s}}}$$

where m_f is equal to the mass of the volume of the fluid occupied by the solid B. If the solid is homogeneous, then $G = C_b$.
The action of the fluid on the body due to the relative motion of the fluid to the body is very more complex. It is equivalent to a torsor $T_{f,\mathbf{s}} = T_{f,\mathbf{s}}(\mathbf{s}, \mathbb{V}, \mathbf{\Gamma})$. The dependency with $\mathbf{\Gamma}$ leads to inertial effects known as added mass effects. This effect cannot be neglected for example for more than air light airships with a big contact surface. Analytic calculations of added mass and added inertia matrix are complicated and cannot be done excepted for specific geometry of the body. In first estimate we may suppose that $T_{f,\mathbf{s}} = T_{f,\mathbf{s}}(\mathbf{s}, \mathbb{V})$.
Suppose that the wind speed $\vec{V}_w$ is constant, uniform and homogeneous in the space $\mathcal{E}$. (almost for a sufficient large domain $U \in \mathcal{E}$). That means that $\vec{v}_p = \vec{V}_w$ for all fluid particle p occupying any position in U, this speed being calculated with respect to a frame rigidly attached to the earth. This hypothesis is very strong all the more so as this is supposed to remain valid even when the body B is moving in U. That assumes that the fluid motion is not (too much!) perturbed by the presence and the motion of the body. That is obviously false in a neighborhood of the contact surface between the fluid and the body. (If $\vec{V}_w = \vec{V}_w(t)$ is not constant over time, the following analysis has to be done at each t. However, the physical assumption is still less reasonable.) Our goal is to get models for command and control of flying objects. Without such strong hypotheses about the fluid motion, we then have to calculate the fluid motion which is totally out of the scope of this curse and in opposite from what needs any automation expert. Remark, that in these conditions, a frame attached to the wind remains an inertial frame if the frame rigidly attached to the earth is itself an inertial frame.
For all particle $p \in B$, in each configuration $\mathbf{s}$ let $\mathbb{V}_r(M_p) = \mathbb{V}(M_p) - \vec{V}_w$ the relative velocity of the particle p with respect to the fluid. On an infinitesimal surface dS_p, the fluid puts a pressure so that the whole torsor is a simple force $C_a, \vec{f_a}$ where C_a is indifferently called (in this curse) the aerodynamic center or center of aerodynamic thrust. The angle between $\mathbb{V}(C_{a,\mathbf{s}})$ (or $\mathbb{V}(G_\mathbf{s})$) and $\mathbb{V}_r(C_{a,\mathbf{s}})$ (or $\mathbb{V}_r(G_\mathbf{s})$) is called the attack angle. The vector $\vec{f_a}$ of the aerodynamic thrust is decomposed in two components: the one parallel

(and opposite) to the direction $\vec{d}$ of $\mathbb{V}_r(C_{a,\mathbf{s}})$ (or $\mathbb{V}_r(G_\mathbf{s})$) called the (aerodynamic) drag noted $\vec{f}_{a,d}$ and the one perpendicular $\vec{\ell}$ to the direction of $\mathbb{V}_r(C_{a,\mathbf{s}})$ (or $\mathbb{V}_r(G_\mathbf{s})$) called the (aerodynamic) lift noted $\vec{f}_{a,\ell}$. Only experimental investigations allow to get mathematical expressions of $\vec{f}_{a,d}$ and $\vec{f}_{a,\ell}$. They are depending on the square of the norm of $\mathbb{V}_r(C_{a,\mathbf{s}})$ (or $\mathbb{V}_r(G_\mathbf{s})$), on the volume mass of the fluid ρ_f (that also depends on the fluid pressure, the temperature, the altitude), on the contact surface S of the solid with the fluid and on a coefficient (drag and lift coefficients C_d and C_ℓ) characteristic of the shape of the body. Generally, the rotation of the body is neglected in this analysis and $\mathbb{V}_r(C_{a,\mathbf{s}})$ and $\mathbb{V}_r(G_\mathbf{s})$ are then considered equal. Finally, we will write:

$$T_{f,\mathbf{s}}(\mathbf{s},\mathbb{V}) = \left\{ \begin{array}{rcl} \omega_{T_{f,\mathbf{s}}(\mathbf{s},\mathbb{V})} & = & \vec{f}_{a,d} + \vec{f}_{a,\ell} = S\rho_f \mid\mid \mathbb{V}_r(C_{a,\mathbf{s}}) \mid\mid^2 (-C_d\vec{d} + C_\ell\vec{\ell}) \\ T_{f,\mathbf{s}}(\mathbf{s},\mathbb{V})(C_{a,\mathbf{s}}) & = & \vec{0} \end{array} \right\}_{C_{a,\mathbf{s}}}$$

5.3 Control forces

Control forces represent mechanical actions due to the operator in a broad sense. Because they are physical actions on a rigid body they may be represented in the whole by a torsor T_c. They depends on a list of independent control parameters usually noted $u = (u_1, \ldots u_p)$ which may be, for example, the intensity of the motor power, direction angles of the thrust of the motors, ... We then may write $T_c = T_c(u) = T_c(u_1, \ldots u_p)$. This wrench is, in a natural way, calculated in a body basis meaning that their components are usually constant in a basis moving with the body or rigidly linked to the body. That is why we write $T_c = T_c(\mathbf{s}, u) = T_c(\mathbf{s}, u_1, \ldots u_p)$. It is impossible to take further the analysis without specifying the system and the control, especially if an eventual nil moment point C_c (for control center) is looked for. Such a point does not always exist.

5.4 How writing the model: assessment

After having described the different possible actions on a body, let us summarize the main issues and their usual solutions for modeling of flying objects. The main benefit in modeling such mechanical systems is that there is no kinematical constraints which leads to an easy and free choice of parameters. The first real issue is the choice of a projection basis to express the vectorial quantities. The usual answer is a body linked basis. The second issue addresses the choice of the point of the space where the wrenches have to be evaluated. We have seen that several points naturally emerge during the modeling process: G, C_b, C_a, C_c. Except for transitory conditions of wind or of control that are very complex to be taking into account, these points and their relative positions are constant in a body frame and may be perfectly described in a reference configuration. Because the inertia center is especially suitable for complex inertial quantities, it is often chosen leading then to the following equations:

$$
\begin{aligned}
m\left.\frac{d\mathbb{V}(G_{\mathbf{s}(t)})}{dt}\right)_{\mathcal{B}} &= m\vec{g} - m_f\vec{g} + \omega_{T_c(\mathbf{s}(t),u)} \\
&\quad + S\rho_f \mid\mid \mathbb{V}_r(C_{a,\mathbf{s}(t)}) \mid\mid^2 (-C_d\vec{d} + C_\ell\vec{\ell}) \qquad (41) \\
\mathbb{I}(\mathbf{s}(t))(\left.\frac{d(\omega_{\mathbb{V}})}{dt}\right)_{\mathcal{B}}) + \omega_{\mathbb{V}} \wedge \mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}}) &= -\overrightarrow{G_{\mathbf{s}(t)}C_{b,\mathbf{s}(t)}} \wedge m_f\vec{g} + T_c(\mathbf{s},u)(G_{\mathbf{s}(t)}) \\
&\quad + \overrightarrow{G_{\mathbf{s}(t)}C_{a,\mathbf{s}(t)}} \wedge \Big(S\rho_f \mid\mid \mathbb{V}_r(C_{a,\mathbf{s}(t)}) \mid\mid^2 (-C_d\vec{d} + C_\ell\vec{\ell})\Big) \qquad (42)
\end{aligned}
$$

Be however aware that derivatives are done with respect to a basis linked to the space frame while, as previously mentioned, the vectors are projected on a body linked basis.This is a main cause of the complexity of the dynamic equations.

6 How changing observation frame

In this section, we investigate the problem of the changing of observation frame. It is an issue of high significance because of the form of the inertial forces according to the nature of observation frame: we know the form of these actions only when the observation frame is a Galilean one. When the "natural" observation frame of a given motion is not a Galilean one, we need to know the motion of this frame with respect to a Galilean one. The aim of this section is then to know how expressing the inertial wrench of a rigid body B with respect to a non Galilean observation frame when its motion with respect to a Galilean observation frame is known. First, we will find the relations of velocity fields, kinetic and dynamic wrenches of a body B when its motion is observed with respect to two observation frame $\mathcal{F}_1$ and $\mathcal{F}_2$. The velocity fields, the kinetic wrenches and the dynamic wrenches of B, when its motion is observed with respect to $\mathcal{F}_i$, are noted $\mathbb{V}_i$, $\mathbb{H}_i$ and $\mathbb{K}_i$ for $i = 1, 2$. Let us remind that $\mathcal{F}_i = (0_i; \mathcal{B}_i)$ for $i = 1, 2$. Time t often is intentionally omitted.

6.1 Composition of velocity fields

In a kinematical point of view, a body is nothing else than a frame. We then may speak about the velocity field $\mathbb{V}_{r,12}$ of $\mathcal{F}_2$ with respect to $\mathcal{F}_1$ whose the sum is $\omega_{\mathbb{V}_{r,12}} = \omega(\mathcal{B}_2/\mathcal{B}_1)$ and whose the value at O_2 is the velocity of O_2 with respect to $\mathcal{F}_1$ namely $\vec{v}(O_2, \mathcal{F}_1) = \left.\frac{d\overrightarrow{O_1O_2}(t)}{dt}\right)_{\mathcal{B}_1}$. We then write:

$$
\mathbb{V}_{r,12} = \left\{ \begin{array}{c} \omega(\mathcal{B}_2/\mathcal{B}_1) \\ \vec{v}(O_2, \mathcal{F}_1) \end{array} \right\}_{O_2}
$$

Thus, the law of composition velocity fields reads:

$$
\mathbb{V}_1 = \mathbb{V}_{r,12} + \mathbb{V}_2 \qquad (43)
$$

□ According to (7), $\omega_{\mathbb{V}_1} = \omega_{\mathbb{V}_{r,12}} + \omega_{\mathbb{V}_2}$. Moreover, if p is a particle of body B and at a configuration $\mathbf{s}(= \mathbf{s}(t))$ such that $M_p = \mathbf{s}(p) = M_p(t)$, then:

$$\begin{aligned}
\mathbb{V}_1(M_p) &= \left.\frac{d\overrightarrow{O_1M_p}(t)}{dt}\right)_{\mathcal{B}_1} \\
&= \left.\frac{d\overrightarrow{O_1O_2}(t)}{dt}\right)_{\mathcal{B}_1} + \left.\frac{d\overrightarrow{O_2M_p}(t)}{dt}\right)_{\mathcal{B}_1} \\
&= \left.\frac{d\overrightarrow{O_1O_2}(t)}{dt}\right)_{\mathcal{B}_1} + \left.\frac{d\overrightarrow{O_2M_p}(t)}{dt}\right)_{\mathcal{B}_2} + \omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \overrightarrow{O_2M_p(t)} \\
&= \left.\frac{d\overrightarrow{O_2M_p}(t)}{dt}\right)_{\mathcal{B}_2} + \left.\frac{d\overrightarrow{O_1O_2}(t)}{dt}\right)_{\mathcal{B}_1} + \omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \overrightarrow{O_2M_p(t)} \\
&= \mathbb{V}_2(M_p) + \mathbb{V}_{12}(O_2) + \omega_{\mathbb{V}_{r,12}} \wedge \overrightarrow{O_2M_p} \\
&= \mathbb{V}_2(M_p) + \mathbb{V}_{r,12}(M_p)
\end{aligned}$$

■

$\mathbb{V}_{12}(M_p)$ may also be understood as the velocity of the particle p as if the body was at rest with respect to $\mathcal{F}_2$.

6.2 Composition of acceleration.Case of a particle.

We start the calculation of the composition of accelerations for a particle p. We successively get:

$$\begin{aligned}
\vec{\gamma}(p, \mathcal{F}_1) &= \left.\frac{d^2\overrightarrow{O_1M_p}(t)}{dt^2}\right)_{\mathcal{B}_1} \\
&= \frac{d}{dt}\left[\left.\frac{d\overrightarrow{O_2M_p}(t)}{dt}\right)_{\mathcal{B}_2} + \left.\frac{d\overrightarrow{O_1O_2}(t)}{dt}\right)_{\mathcal{B}_1} + \omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \overrightarrow{O_2M_p(t)}\right]\Bigg)_{\mathcal{B}_1} \\
&= \frac{d}{dt}\left[\left.\frac{d\overrightarrow{O_1O_2}(t)}{dt}\right)_{\mathcal{B}_1} + \omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \overrightarrow{O_2M_p(t)}\right]\Bigg)_{\mathcal{B}_1} + \frac{d}{dt}\left[\left.\frac{d\overrightarrow{O_2M_p}(t)}{dt}\right)_{\mathcal{B}_2}\right]\Bigg)_{\mathcal{B}_1} \\
&= \frac{d}{dt}\left[\left.\frac{d\overrightarrow{O_1O_2}(t)}{dt}\right)_{\mathcal{B}_1} + \omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \overrightarrow{O_2M_p(t)}\right]\Bigg)_{\mathcal{B}_1} + \\
&\quad \frac{d}{dt}\left[\left.\frac{d\overrightarrow{O_2M_p}(t)}{dt}\right)_{\mathcal{B}_2}\right]\Bigg)_{\mathcal{B}_2} + \omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \left.\frac{d\overrightarrow{O_2M_p}(t)}{dt}\right)_{\mathcal{B}_2} \\
&= \vec{\gamma}_{r,12}(p) + \vec{\gamma}_{c,12}(p) + \vec{\gamma}(p, \mathcal{F}_2)
\end{aligned}$$

with

- $\vec{\gamma}(p,\mathcal{F}_2) = \left.\dfrac{d^2\overrightarrow{O_2M_p}(t)}{dt^2}\right)_{\mathcal{B}_2}$ the acceleration of p with respect to $\mathcal{F}_2$
- $\vec{\gamma}_{c,12}(p) = 2\omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \left.\dfrac{d\overrightarrow{O_2M_p}(t)}{dt}\right)_{\mathcal{B}_2} = 2\omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \vec{v}(p,\mathcal{F}_2)$ the so-called Coriolis acceleration of p which is appearing when the particle is moving with respect to a frame which is itself rotating.
- $\vec{\gamma}_{r,12}(p) = \left.\dfrac{d^2\overrightarrow{O_1O_2(t)}}{dt^2}\right)_{\mathcal{B}_1} + \left.\dfrac{d\omega(\mathcal{B}_2/\mathcal{B}_1)}{dt}\right)_{\mathcal{B}_1} \wedge \overrightarrow{O_2M_p(t)} + \omega(\mathcal{B}_2/\mathcal{B}_1) \wedge (\omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \overrightarrow{O_2M_p(t)})$ which is the acceleration of the particle p as if the body was at rest with respect to $\mathcal{F}_2$ and which is not the derivative of $\vec{v}_{r,12}(p) = \mathbb{V}_{r,12}(M_p)$.
  The model of body as aggregate of particles then allows us to directly transfer the above relations from the particle case to body case by computing integrals. It will be the way to investigate the change of inertia frames for the Dynamic wrench.

6.3 Kinetic wrench and change of frames

We deduce from the rule of the composition of velocity fields (43) that

$$\mathbb{H}_1 = \mathbb{H}_{r,12} + \mathbb{H}_2 \tag{44}$$

namely, if G is the center of mass of B

$$\left\{\begin{array}{c} m\vec{v}(G,\mathcal{F}_1) \\ \mathbb{I}_G(\omega_{B/\mathcal{B}_1}) \end{array}\right\}_G = \left\{\begin{array}{c} m\left[\left.\frac{d\overrightarrow{O_1O_2}(t)}{dt}\right)_{\mathcal{B}_1} + \omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \overrightarrow{O_2G}\right] \\ \mathbb{I}_G(\omega_{\mathcal{B}_2/\mathcal{B}_1}) \end{array}\right\}_G + \left\{\begin{array}{c} m\vec{v}(G,\mathcal{F}_2) \\ \mathbb{I}_G(\omega_{B/\mathcal{B}_2}) \end{array}\right\}_G$$

6.4 Dynamic wrench and change of frames

Unfortunately, the change of frames formulae for the dynamic wrench are not so simple than for the kinetic wrench. By integrating $\vec{\gamma}(p,\mathcal{F}_1) = \vec{\gamma}_{r,12}(p) + \vec{\gamma}_{c,12}(p) + \vec{\gamma}(p,\mathcal{F}_2)$ as for the usual definition of any kinetic wrench namely (the configuration $\mathbf{s}(t)$ is omitted)

$$\mathbb{K}_{r,12}(M) = \int_{p\in B} \overrightarrow{MM_p(t)} \wedge \vec{\gamma}_{r,12}(p)\rho(p)dv(p) \quad \forall M \in \mathcal{E} \tag{45}$$

$$\mathbb{K}_{c,12}(M) = \int_{p\in B} \overrightarrow{MM_p(t)} \wedge \vec{\gamma}_{c,12}(p)\rho(p)dv(p) \quad \forall M \in \mathcal{E} \tag{46}$$

so that

$$\mathbb{K}_1 = \mathbb{K}_{r,12} + \mathbb{K}_{c,12} + \mathbb{K}_2 \tag{47}$$

Computations show that the dynamic wrenches $\mathbb{K}_{r,12}$ and $\mathbb{K}_{c,12}$ also read:

$$\mathbb{K}_{c,12} = \left\{\begin{array}{c} 2m\omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \vec{v}(G,\mathcal{F}_2) \\ \omega_{B/\mathcal{B}_2} \wedge \mathbb{I}_G(\omega_{\mathcal{B}_2/\mathcal{B}_1}) + \omega_{\mathcal{B}_2/\mathcal{B}_1} \wedge \mathbb{I}_G(\omega_{B/\mathcal{B}_2}) + \mathbb{I}_G(\omega_{\mathcal{B}_2/\mathcal{B}_1} \wedge \omega_{B/\mathcal{B}_2}) \end{array}\right\}_G$$

and

$$\mathbb{K}_{r,12} = \left\{ \begin{array}{c} m\vec{\gamma}_{r,12}(G) \\ \mathbb{I}_G(\left.\frac{d\omega(\mathcal{B}_2/\mathcal{B}_1)}{dt}\right)_{\mathcal{B}_1}) + \omega(\mathcal{B}_2/\mathcal{B}_1) \wedge \mathbb{I}_G(\omega(\mathcal{B}_2/\mathcal{B}_1)) \end{array} \right\}_G$$

□ The proof is left to the reader ■

6.5 Dynamic Principle

We may now give the form of the Fundamental Dynamic Principle (FDP) for a motion observed with respect to a non inertial frame $\mathcal{F}_2$ which is itself moving with respect to an inertial frame $\mathcal{F}_1$. The usual exemple is when $\mathcal{F}_2$ is rigidly linked with the earth whose rotation cannot be neglected as for launches of rockets.
Let $T_{ext,\mathbf{s}}$ the wrench of all external actions acting on B in the configuration $\mathbf{s}$ and $T_{in,\mathbf{s},i}$ for $i = 1, 2$ the wrench of the inertial actions due to the motion B with respect to $\mathcal{F}_i$ in the configuration $\mathbf{s}$. The FDP reads $T_{ext,\mathbf{s}} + T_{in,\mathbf{s},i} = 0$ for $i = 1, 2$. Thus, because of Fc_1 is an inertial frame $T_{in,\mathbf{s},1} = -\mathbb{K}_1$. We then deduce the FDP for the motion observed with respect to the non inertial frame $\mathcal{F}_2$:

$$\mathbb{K}_2 = T_{ext,\mathbf{s}} - \mathbb{K}_{r,12} - \mathbb{K}_{c,12}$$

It is the reason why the complementary terms $\mathbb{K}_{r,12} + \mathbb{K}_{c,12}$ are sometimes called the complementary forces appearing as if they were real forces acting on B like $T_{ext,\mathbf{s}}$.

7 Systems of rigid bodies

7.1 First Newton Law

Suppose now we have two rigid bodies B_i and B_j. It may be exist a kinematic link or not between the two bodies (you may think about the gravitational mutual action between the moon and the earth) which will be investigated in the following paragraph. Independently of these kinematic features, suppose that there is an action of B_i on B_j represented in a given configuration by a wrench T_{ji}. Then the first Newton law claims that the wrench T_{ij} of the action of B_j on B_i is such that:

$$T_{ij} + T_{ji} = 0 \qquad (48)$$

and this does not depend on the nature of the observation frame namely in any frame.

7.2 Kinematical links

In systems of rigid bodies like in robots (terrestrial or/and aerial), there are physical systems called joints or kinematical joints or mechanical joints that constrain the relative motions of two linked bodies B_i and B_j linked by the link $\mathcal{L}_{i,j}$ so that the degree of freedom of the motion of the system $\Sigma = \{B_i, B_j\}$ is not 6+6 = 12 but less than 12. In practice, there is a great variety of mechanical joints for realizing the kinematical links but we provide only three of the most usual.

- a rotational joint means that the relative motions are rotations about an axis $\Delta_{i,j}$
- a spherical joint means that the relative motions are rotations about a point $O_{i,j}$
- a translational or prismatic joint means that the relative motions are translations along an axis $\Delta_{i,j}$

You have to pay attention to the fact that $\Delta_{i,j}$ and $O_{i,j}$ are moving with $\Sigma = \{B_i, B_j\}$ and then depends on the configuration $\mathbf{s} = (\mathbf{s}_i, \mathbf{s}_j)$ of Σ. Rotational and translational joints are one degree of freedom links whereas spherical joints are three degrees of freedom links. That means that, for the formers, 5 degrees of freedom are constrained without motion with respect to B_i and/or B_j whereas only three degrees of freedom are constrained for the latter.
A big part of the art of mechanical modelling consists in well choosing the parameters or variables describing the system in order to take into account all the kinematical joints of the mechanical system. If the graph of the links of the system is a tree without loop so that you may choose a body B_0 as a root of the system and a unique so-called kinematical chain γ_i (namely a path on the graph) from the root B_0 to B_i for all body B_i of the system, a usual way to parametrize the motion of each body B_i consists in:

- first parametrizing the (absolute) motion of the root-body or base-body B_0. For aerial robots, this motion is a free motion in the space with 6 degrees of freedom.
- second parametrizing the relative motion of each B_i with respect to B_0 thanks to the unique kinematical chain γ_i between B_0 and B_i

7.3 The wrench $T_{\ell_{ij}}$ associated with a mechanical link $\mathcal{L}_{i,j}$

Thanks to the first Newton law the dynamical features of the mechanical link $\mathcal{L}_{i,j}$ are described by the wrench T_{ij} or by its opposite T_{ji}. We decide here to choose $T_{\ell_{ij}} = T_{ij}$.
It may be prove that, after applying the first Newton Law for all mechanical links of the system and the second Newton law for each body of the system, there are more unknowns than equations: the principles of mechanics are not enough to have a closed system of equations and additional equations are necessary to form a closed system of equations. These additional equations are called constitutive equations or equations of constitutives laws of the system. They describe the mechanical behavior of the joints of the system. The form of these laws is postulated and the terms of these laws are definitively obtained thanks to experiments.
This mechanical or dynamical aspect of a joint must not be confused with the kinematic aspect described above. Another distinction must be done. Sometimes one speaks of active joint and passive joint. The term active joint refers to when motors are placed so that they induce a relative motion of a body with respect to another body. These active terms are not concerned by the constitutive laws which address only the passive features. Nevertheless, it is highly significant to know what is controlling the motor. If it is controlling the motion(translational or rotational), then the corresponding motoring action (force or torque) is unknown but when the motor delivers a known mechanical action (force or torque) then the corresponding motion is unknown. It is the main duality principle of mechanics:
IT IS IMPOSSIBLE TO PRESCRIBE (NAMELY TO KNOW A PRIORI) TOGETHER A MOTION AND THE ACTION LEADING THIS MOTION.

The following developments only concern the passive features of a joint. Contrary to external actions like gravitation, buoyancy or aerodynamic actions, mechanical actions attached to a mechanical link $\mathcal{L}_{i,j}$ are such that the wrench $T_{\ell_{ij}}$ is unknown. Taken into account the constitutive law of the joint leads to give informations about A PART of $T_{\ell_{ij}}$ and ONLY A PART, the rest of $T_{\ell_{ij}}$ remaining unknown. There is a large zoology of the types of joint and we give hereafter only three of them which are the most usual in aerial robots. Remark that for students who already investigated modelings of cars, the big difficulty in this case deals with the modeling of contact between the car and the road. The corresponding kinematical joint is a so -called non holonomic link and the constitutive law is often a so-called Coulomb friction law: this type of link induce deep difficulties in the modeling. Fortunately, in the space, for aerial robots, such difficulties cannot happen. The main difficulty in aerial robotic is the spatial feature of the motion. The three main types of links are

- the perfect link
- the (linear) elastic link
- the (linear) viscous damper link

For each kinematical joint we give the corresponding wrench $T_{\ell_{ij}}$ only for perfect link which is the main type of (passive) joint met in models of aerial robots. For the rotational and translational links we fix a point A_i which belongs to $\Delta_{i,j}$ and which is linked to B_i. The basis for projecting the vectors is linked to the body B_i so that $\Delta_{i,j} = (A_i; \vec{k})$. Thus (obviously in any configuration):

- for a rotational joint:

$$T_{\ell_{ij}} = \left\{ \begin{array}{rcl} \omega_{T_{\ell_{ij}}} & = & \vec{R} \\ T_{\ell_{ij}}(A_i) & = & \vec{M} \end{array} \right\}_{A_i}$$

  where $\vec{R}$ is any and $\vec{M}$ is any such that $(\vec{M} \mid \vec{k}) = 0$.

- for a spherical joint:

$$T_{\ell_{ij}} = \left\{ \begin{array}{rcl} \omega_{T_{\ell_{ij}}} & = & \vec{R} \\ T_{\ell_{ij}}(O_{i,j}) & = & \vec{0} \end{array} \right\}_{O_{i,j}}$$

  where $\vec{R}$ is any

- for a translational or prismatic joint

$$T_{\ell_{ij}} = \left\{ \begin{array}{rcl} \omega_{T_{\ell_{ij}}} & = & \vec{R} \\ T_{\ell_{ij}}(A_i) & = & \vec{M} \end{array} \right\}_{A_i}$$

  where $\vec{M}$ is any and $\vec{R}$ is any such that $(\vec{R} \mid \vec{k}) = 0$.

A complete teaching of dynamics of rigid body systems is obviously out of the scope of this short course. When we plan to write the dynamic equations of a mechanical systems two main options are possible. The first one consists in writing the equations of the Fundamental Principle of Dynamics for each body of the system and to add the additional Action-Reaction Principle sometimes called the first Newton's law. Generally, the equations deduced from these Principles of Mechanics provide less equations than the variables necessary for describing the evolution of the system namely for writing the dynamical system whose solution gives the trajectory of the system. We call these equations the equations of motion of the system. The additional necessary equations or relations must specify the mechanical behavior of the links between the bodies of the system. They are called the constitutive relations of the mechanical system. This Newton-Euler approach is however complicated especially because it leads to produce more equations than necessary for the motion problem and involve more unknowns that those necessary for our initial goal.
The second approach proposed for the first time by Lagrange in its book "Mécanique Analytique" is better adapted for our modeling objective. It will produce the motion equation and will automatically eliminate the supplementary unknowns that are without interest for the motion problem. We then will systematically use it.

7.4 Parametrization of rigid body system

This is the first step of the modeling and this is the most important. A wrong parametrization leads to wrong equations even if the principles and the calculations are right. A right but not well appropriate choice of variables describing the system may lead to inextricable equations of motion.
Let $\Sigma = \{S_1, \dots, S_m\}$ a system of m bodies. Suppose that these bodies are subjected to kinematic constraints because of the links between the bodies. If there are p such kinematic constraints in the system, then there are $6m - p$ independent variables. Suppose that these relations may be written as $f_i(x) = 0$ for $i = 1, \dots, p$ and that the family $x = (x_i^j)_{1 \le j \le 6, 1 \le i \le m}$ of 6m variables describe the configurations of the system without constraint (the 6 variables $(x_i^j)_{1 \le j \le 6}$ describe the configurations of the free body S_i). Such a system is called holonomic because the kinematic constraints do not involve the velocities of the bodies S_i. For aerial robots there are a priori only holonomic constraints, non holonomic mechanical constraints involving generally sliding motions. This is one of the great differences between aerial and terrestrial vehicle motions.
Thanks to the implicit function theorem, the relations $f_i(x) = 0$ allow to exhibit (at least locally) a family $q = (q_1, \dots, q_n)$ of parameters (called Lagrange coordinate of Σ) so that the position M_P of any particle P of the system Σ in any configuration is $M_P = M_P(t, q_1, \dots, q_n) = M_P(t, q)$. The explicit dependency on time t ww<qarises when a part of the motion of Σ is prescribed by the control of the system.

7.5 Kinematics of the system Σ

The kinematics of the system is described by the velocity vector fields $\mathbb{V}_i = \mathbb{V}_i(t, q, \dot{q})$ for each body S_i of Σ. We write $\mathcal{V} = \mathcal{V}(t, q, \dot{q}) = (\mathbb{V}_i(t, q, \dot{q}))_{i=1,\dots,n}$ which defines a vector field on the set of configurations at (t, q). For all $i = 1, \dots, n$, $\mathbb{V}_i$ always reads:

$$\mathbb{V}_i(t, q, \dot{q}) = \mathbb{W}_i(t, q) + \sum_k^n \dot{q}_k \mathbb{W}_i^k(t, q) \tag{49}$$

A virtual velocity field on Σ in the configuration (t, q) is a vector field on the set of configurations at (t, q) defined by $\mathcal{V}^* = \mathcal{V}^*(t, q) = (\mathbb{V}_i^*(t, q))_{i=1,\ldots,n}$ where

$$\mathbb{V}_i^*(t, q, \dot{q}) = \sum_k^n q_k^* \mathbb{W}_i^k(t, q) \tag{50}$$

and you may note the difference between (49) and (50). In this last expression, $q^* = (q_k^*)_{k=1,\ldots,n}$ is any vector of $\mathbb{R}^n$.

7.6 Kinetics of the system Σ

The kinetics of the system is entirely encoded in the total kinetic energy $C(t, q, \dot{q})$ which is the sum of the kinetic energy of each body S_i. Suppose the motion of the system Σ is observed with respect to an inertial frame $\mathcal{F}$ and let $\mathbb{H}_i$ be the kinetic wrench or kinetic torsor of S_i for all$i = 1, \ldots, m$. Then:

$$C(t, q, \dot{q}) = \frac{1}{2} \sum_{i=1}^m [\mathbb{H}_i | \mathbb{V}_i] \tag{51}$$

7.7 Virtual power of a given action

Suppose a given action ϕ whose action on each body S_i may be described by a wrench $T_{\phi,i} = T_{\phi,i}(t, q, \dot{q}$ because the body is rigid. By definition, in the configuration (t, q) the virtual power of ϕ on the full system Σ is:

$$\mathcal{P}_\phi^* = \sum_{i=1}^m [T_{\phi,i}(t, q, \dot{q} | \mathbb{V}_i^*] \tag{52}$$

By use of (50), (52) reads:

$$\mathcal{P}_\phi^* = \sum_{k=1}^m Q_{\phi,k} q_k^* \tag{53}$$

with $Q_{\phi,k} = Q_{\phi,k}(t, q, \dot{q}) = \sum_{i=1}^m [T_{\phi,i}(t, q, \dot{q}) | \mathbb{W}_i^k(t, q)]$ for all $k = 1, \ldots, n$. It is called the factor power for parameter q_k relatively to the action ϕ.

For a family of given action $\phi_1, \phi_2, \ldots$, one performs the sum of the corresponding factor powers:

$$\mathcal{P}_g^* = \sum_{k=1}^n Q_k(t, q, \dot{q}) q_k^* \tag{54}$$

with $Q_k(t, q, \dot{q}) = \sum_{\phi \in \{\phi_1, \phi_2, \ldots\}} Q_{\phi,k}(t, q, \dot{q})$.

7.8 Virtual power of a acceleration quantities

When the action is produced by the acceleration quantities on each particle of the system, then there is the beautiful Lagrange's formula

Theorem 10

$$\mathcal{P}^*_{acc} = \sum_{i=1}^{m} [\mathbb{K}_i(t, q, \dot{q}, \ddot{q}) | \mathbb{V}^*_i] = \sum_{k=1}^{n} J_k q^*_k \tag{55}$$

with

$$J_k = J_k(t, q, \dot{q}, \ddot{q}) = \frac{d}{dt}\frac{\partial C}{\partial \dot{q}_k} - \frac{\partial C}{\partial q_k} \tag{56}$$

7.9 Virtual power of link actions

For an aerial robot Σ, the link actions are the internal actions between the bodies of Σ that are linked by a kinematic joint or articulation. For robots, the three main kinds of articulations are prismatic joints, rotational joints and revolute joints. A prismatic joint lets free a translation motion along an axis, the rotational joint lets free a rotation motion about an axis and a revolute or spherical joint lets free a rotation about a point. The two first are one degree of freedom joint whereas the third is a three degree of freedom joint. By definition, if the kinematic joint called $\mathcal{L}_{ij}$ takes place between the solids S_i and S_j, we already stressed that a part of the torsor T_{ij} of the action of S_j on S_i is unknown (namely must be calculated) whereas the other part is given for example as a function of variables $t, q, \dot{q}$. This is the main difference between the given actions and the link actions. By definition, the virtual power of the link $\mathcal{L}_{ij}$ is:

$$\mathcal{P}^*_{\mathcal{L}_{ij}} = [T_{ij} | \mathbb{V}^*_i - \mathbb{V}^*_j] \tag{57}$$

which, thanks to the action-reaction law $T_{ij} + T_{ji} = 0$, does not depend on the orientation of the link:

$$[T_{ij} | \mathbb{V}^*_i - \mathbb{V}^*_j] = [T_{ji} | \mathbb{V}^*_j - \mathbb{V}^*_i]$$

By use of Virtual Velocities Power, the unknown parts of the link torsors are removed from the equations even if obviously they do not vanish. The given part of the link actions is given through a constitutive law of this link. The most usual constitutive laws are the perfect joints and the elastic joints.
If $\mathcal{L}_{ij}$ is perfect, then $\mathcal{P}^*_{\mathcal{L}_{ij}} = 0$. If $\mathcal{L}_{ij}$ is elastic, there is a function $U_{\mathcal{L}_{ij}} : (t, q) \mapsto U_{\mathcal{L}_{ij}}(t, q)$ called the elastic potential of the link such that:

$$\mathcal{P}^*_{\mathcal{L}_{ij}} = -\sum_{k=1}^{n} \frac{\partial U_{\mathcal{L}_{ij}}}{\partial q_k} q^*_k \tag{58}$$

From now on, we suppose that all the links are perfect or elastic and U_ℓ is the sum of all the elastic potential of the elastic links. Then, the virtual power of link actions reads

$$\mathcal{P}^*_\ell = -\sum_{k=1}^{n} \frac{\partial U_\ell}{\partial q_k} q^*_k \tag{59}$$

7.10 Principle of Virtual Power (PVP). Lagrange's equations

According to the PVP, if the motion is observed with respect to a Galilean frame, then at each t and for all VVF $\mathcal{V}^*$ at $(t, q(t))$:

$$\mathcal{P}^*_{acc} = \mathcal{P}^*_\ell + \mathcal{P}^*_g$$

or

$$\sum_{k=1}^{n}\left(\frac{d}{dt}\frac{\partial C(t,q(t),\dot{q}(t))}{\partial \dot{q}_k} - \frac{\partial C(t,q(t),\dot{q}(t))}{\partial q_k}\right)q^*_k = \sum_{k=1}^{n}\left(Q_k(t,q(t),\dot{q}(t)) - \frac{\partial U_\ell(t,q(t))}{\partial q_k}\right)q^*_k$$

Because the quantities q^*_k are any, we deduce the n following Lagrange's equations:

$$\frac{d}{dt}\frac{\partial C(t,q(t),\dot{q}(t))}{\partial \dot{q}_k} - \frac{\partial C(t,q(t),\dot{q}(t))}{\partial q_k} = Q_k(t,q(t),\dot{q}(t)) - \frac{\partial U_\ell(t,q(t))}{\partial q_k} \ \ \forall k = 1,\dots,n \tag{60}$$

The dynamic equations of Σ set up a nonlinear dynamical system of order 2 and size n. As illustrated in the web link suggested in the introduction, the simulation for the very simple planar triple pendulum shows that these dynamical systems are chaotic. A rational use of these equations is then double

8 Linearization: vibration analysis

8.1 Generalities

Suppose now that there is an equilibrium position of Σ defined by q_e which means that the constant function $t \mapsto q_e$ is solution of (60). q_e is then solution of the following equilibrium equations:

$$Q_k(t,q_e,0) - \frac{\partial U_\ell(t,q_e)}{\partial q_k} = 0 \ \ \forall k = 1,\dots,n \tag{61}$$

Let $X = q - q_e$ the element describing the gap with the equilibrium configuration. In this section, we suppose that X is small which means nothing *per se* but which means that $||X||$ is small with respect to a characteristic dimension of Σ. We also suppose that successive time derivatives of X have same order of magnitude than X.
Then the linearization of (60) consists in keeping only first order terms in the variables $X, \dot{X}, \ddot{X}$ in (60) with $q = q_e + X$. By using (61), the linearized equations then read with the matrix form:

$$M(q_e)\ddot{X} + D(q_e)\dot{X} + K(q_e)X = 0 \tag{62}$$

where M is the mass matrix, D is the damping matrix and K is the stiffness matrix (at q_e). These matrices are element of $\mathcal{M}_n(\mathbb{R})$. M is symmetric positive definite, K and D are any except when the given forces are conservative and derived from a potential $U_g(t,q)$. In this case, $D = 0$ and $K_{ij}(q_e) = \dfrac{\partial^2 U(q_e)}{\partial q_i \partial q_j}$ with $U = U_g + U_\ell$ and K is then symmetric. The stability of q_e is directly linked with the definitiveness of $K(q_e)$.

There exists a theory of control of vibrations but *a priori* for aerial robots the human beings only control the finite motion of the system and not its vibrations. This system undergoes passively the vibrations that

are then out of control. Often, there are natural limitations of the vibrations of the system which may be ignored in the model except for three situations which may lead to the destabilization and to the ruin of the system.
These three main modes of instability are called divergence, flutter and resonance. We quickly explain these situations. Suppose now that $K = K(q_e, c)$ where c is a parameter. For example it is the speed of the wind. According to the eventual symmetry of $K(q_e, c)$(existence or not for the external given actions), the situation is dramatically different. Let S the square root of M which exists because M is symmetric. Let $\tilde{K}(q_e, c) = S^{-1}K(q_e, c)S^{-1}$ be a matrix that contains mixed informations about M and $K(q_e, c)$. $\tilde{K}(q_e, c)$ is symmetric if and only $K(q_e, c)$ is symmetric. The eigenvalues of $\tilde{K}(q_e, c)$ are generally real positive for $c = 0$ which corresponds to a pure elastic system. Because for $c = 0$ the system is supposed conservative stable, they are also strictly positive.

Definition 7 *The eigenvalues of $\tilde{K}(q_e, c)$ are also the roots of the generalized characteristic polynomial $P(\lambda) = \det(K(q_e, c) - \lambda M)$. The positive square roots ω_k of the n eignevalues λ_k are called the eigenfrequencies or natural frequencies of the system. The generalized eigenvectors X_k for $k = 1, \ldots, n$ are solutions of*

$$K(q_e, c)X_k = \omega_k^2 M X_k$$

Obviously, $\omega_k = \omega_k(q_e, c)$ and $X_k = X_k(q_e, c)$ for $k = 1, \ldots, n$. For conservative stable systems, the equations may be decoupled by projecting it on the modal basis $(X_k)_{k=1,\ldots,n}$. If the kinematic unknown X is calculated on this basis, then $X(t) = \sum_{k=1} r_k(t)X_k$ and the equation (62) reads

$$\ddot{r}_k(t) + \omega_k^2 r_k(t) = 0 \quad \forall k = 1, \ldots, n \tag{63}$$

after having used the fundamental relations $X_k^T K X_\ell = X_k^T M X_\ell = 0$ if $k \neq \ell$

8.2 Divergence instability

Definition 8 *Divergence instability occurs for the value c_d^* of the parameter such that the smallest eigenfrequency vanishes. This is also the value that makes singular the stiffness matrix:*

$$\omega_1(q_e, c_d^*) = 0 \quad or \quad \det K(q_e, c_d^*) = 0 \tag{64}$$

8.3 Flutter instability

Definition 9 *Flutter instability occurs for the value c_f^* of the parameter such that $\tilde{K}(q_e, c_f^*)$ ceases to be diagonalizable. This is often occurring when two simple eigenfrequencies, for example $\omega_1(q_e, c) < \omega_2(q_e, c)$ distinct for $c < c_f^*$ become equal for $c = c_f^*$ or equivalently when the discriminant $\Delta(q_e, c)$ of the characteristic polynomial $P(\lambda)$ vanishes for $c = c_f^*$:*

$$\omega_1(q_e, c_f^*) = \omega_2(q_e, c_f^*) \quad or \quad \Delta(q_e, c_f^*) = 0 \tag{65}$$

This instability mode is specific to the wind action on a wing or to the engine thrust on a rocket (see the last exercice of the tutorials)

8.4 Resonance instability

The resonance instability occurs when a mechanical system is excited by a action $\phi(t)$ periodic with period T such that a multiple of $\omega = \frac{2\pi}{T}$ is close to one eigenfrequency ω_k of the free system.This phenomenon is purely dynamic.
(62) becomes:

$$M(q_e)\ddot{X} + D(q_e)\dot{X} + K(q_e)X = \phi(t)Y \tag{66}$$

so that (63) becomes

$$\ddot{r}_k(t) + \omega_k^2 r_k(t) = a_k\phi(t) \quad \forall k = 1, \dots, n \tag{67}$$

By expanding $\phi(t)$ and $r_k(t)$ into a Fourier series, we deduce that for all integer $p \geq 1$

$$\ddot{r}_{k,p}(t) + \omega_k^2 r_{k,p}(t) = b_{k,p}\sin(p\omega t) \quad \forall k = 1, \dots, n \tag{68}$$

It is well-known that- up to a solution of the homogeneous equation- a solution of (68) is $\frac{b_{k,p}}{\omega_k^2 - p^2\omega^2}\sin(p\omega t)$ whose magnitude increases more and more up to infinity when $p\omega$ is closer and closer to the natural eigen-frequency ω_k.

9 Tutorials

Exercice 1 *Show that the set of squares matrices $\mathfrak{M}_n(\mathbb{R})$ is a Lie algebra with the commutator*

$$[M, N] = MN - NM$$

as Lie bracket.

Exercice 2 *Calculate the exponential of the skew symmetric matrix*

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

What do you observe?

Exercice 3 *Show that $\mathfrak{D}(\mathcal{E})$ equipped with the Lie bracket*

$$[X, Y](M) = \omega_X \wedge Y(M) - \omega_Y \wedge X(M) \quad \forall M \in \mathcal{E} \tag{69}$$

defined in the course is a 6-dimensional Lie algebra.

Exercice 4 *With the notations of the course (see figure 2), for the Yaw (ψ)-Pitch (θ)-Roll (ϕ) angles representation of rotations, calculate* **P** *as function of parameters ψ, θ, ϕ. Calculate the angular velocity as a function of these parameters and their derivative in $\mathcal{B}$ and in $\mathcal{B}'$.*

Exercice 5 *Prove that if $X, Y \in \mathfrak{D}(\mathcal{E})$, $[X \mid Y] = (\omega_X \mid Y(M)) + (\omega_Y \mid X(M))$ has a right-hand side that is independent on $M \in \mathcal{E}$ and only depends on X and Y. Prove then that $[. \mid .]$ is a symmetric non degenerate bilinear form on $\mathfrak{D}(\mathcal{E})$ but that is not a scalar product on $\mathfrak{D}(\mathcal{E})$.*

Exercice 6 *Calculate the inertia operators $\mathbb{I}_G$ of the following rigid bodies at their inertia center G for :a parallelepiped a, b, c, a rectangle a, b, a cylinder R, h, a bar h. The bodies are homogeneous. Their mass is m. If necessary, you may use an appropriate density of mass repartition.*

Exercice 7 *Prove that for a rigid body, the kinetic energy reads:*

$$C_{\mathbf{s}(t)} = \frac{1}{2}[\mathbb{H} \mid \mathbb{V}] = \frac{1}{2}m\mathbb{V}(G_{\mathbf{s}(t)})^2 + \frac{1}{2}(\mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}}) \mid \omega_{\mathbb{V}}) \tag{70}$$

Exercice 8 *Let S a bar (AB) drawn on the figure 3 with length 2ℓ, homogeneous with mass m. The motion is observed with respect to an inertial frame identified with the orthonormal coordinate system $\mathcal{R} = (O; \vec{i}, \vec{j}, \vec{k}) = (O; \mathcal{B})$ as on the figure 2. S is subjected to two given physical actions: a force $\vec{F_B}(t) = -F_B(t)\frac{\vec{AB}}{||\vec{AB}||}$ modeling the command of S acting at B and the gravity. θ is the angle $(\widehat{\vec{i}, \vec{AB}}) = (\widehat{\vec{i}, \vec{i}_S})$ and y the second coordinate of A.*

1. *Calculate the coordinates of G and B in $\mathcal{R}$ as function of y, θ and of the parameter ℓ.*
2. *Is y, θ a parametrization of the motion of the body?*
3. *Calculate the velocity field V of S as a function of $y, \theta, \dot{y}, \dot{\theta}$.*
4. *Show that the inertia matrix $\mathcal{J}_G(\mathcal{B}_S)$ of S at G in $\mathcal{B}_S = (\vec{i}_S, \vec{j}_S, \vec{k})$ reads*

$$\mathcal{J}_G(\mathcal{B}_S) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

5. *Calculate I. Use if necessary, $\rho = \frac{m}{2\ell}$ the linear mass density of S.*
6. *Calculate the kinetic wrench $\mathbb{H}$ of S.*
7. *Calculate the kinetic energy of the body S.*
8. *Calculate the equivalent wrenches $\mathbf{T}_{F_B}$ and $\mathbf{T}_g$ of the forces $\vec{F_B}$ and of the gravity $\vec{g} = g\vec{i}$.*
9. *Is there another external action acting on B.*
10. *Find the dynamic equations of S.*
11. *Is the angle θ controllable by the single force $\vec{F_B}(t)$?*
12. *Do again the same exercise when the motion $t \mapsto y(t)$ is a given function (for example $y(t) = a \sin \omega t$).*

Exercice 9 *Let S a square body (A, B, C, D) drawn on the figure 4 with side 2ℓ, homogeneous with mass m The motion is observed with respect to an inertial frame identified with the orthonormal coordinate system $\mathcal{R} = (O, \vec{i}, \vec{j}, \vec{k}) = (O; \mathcal{B})$ as on the figure 3. S is subjected to two forces $\vec{F_A}(t) = F_A(t)\vec{j}_S$ and $\vec{F_D}(t) = F_D(t)\vec{j}_S$ (command of S) acting at A and at D and a force $\vec{F_T}$ (modeling the drag) opposite to the motion and acting at C_a middle of $[B, C]$. (x, y, θ) is chosen as a parametrization of the motion, where (x, y) are the coordinates of the inertia center G and θ is the angle $(\widehat{\vec{i}, \vec{AD}}) = (\widehat{\vec{i}, \vec{i}_S})$.*

1. *Calculate the coordinates of A and D in $\mathcal{R}$ as function of x, y, θ and of the parameter ℓ.*
2. *Calculate the velocity fiel V of S .*
3. *Show that the inertia matrix $\mathcal{J}_G(\mathcal{B}_S)$ of S at G in $\mathcal{B}_S = (\vec{i}_S, \vec{j}_S, \vec{k})$ reads*

$$\mathcal{J}_G(\mathcal{B}_S) = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

4. *Calculate I_3. Use if necessary, $\sigma = \frac{m}{2\ell}$ the areal mass density of S.*
5. *Calculate the kinetic wrench $\mathbb{H}$ of S.*
6. *Calculate the kinetic energy of the body S.*
7. *Calculate the equivalent torsors $\mathbf{T}_{F_A}$ and $\mathbf{T}_{F_D}$ of the forces $\vec{F}_A$ and $\vec{F}_D$.*
8. *Same question with the equivalent torsor $\mathbf{T}_{F_T}$ of $\vec{F}_T$*
9. *Find the dynamic equations of S.*
10. *Is the angle θ controllable by the single force $\vec{F_A}(t)$?*

Exercice 10 *Prove that for a rigid body, the kinetic energy reads:*

$$C_{\mathbf{s}(t)} = \frac{1}{2}[\mathbb{H} \mid \mathbb{V}] = \frac{1}{2}m\mathbb{V}(G_{\mathbf{s}(t)})^2 + \frac{1}{2}(\mathbb{I}(\mathbf{s}(t))(\omega_{\mathbb{V}}) \mid \omega_{\mathbb{V}}) \tag{71}$$

Exercice 11 *Let $\Sigma = (S_1, S_2)$ a system made up of two bars $S_1 = (AB)$ and $S_2 = (BC)$ drawn on the figure 5. S_1 and S_2 have same length 2ℓ and are homogeneous with mass m. G_i is the center of mass of S_i for $i = 1, 2$. The motion is observed with respect to an inertial frame identified with the orthonormal coordinate system $\mathcal{R} = (O; \vec{i}, \vec{j}, \vec{k}) = (O; \mathcal{B})$ as on the figure 4. S_1 is subjected to two given physical actions: a force $\vec{F_A} = u\vec{i} + v\vec{j}$ modeling the command of S_1 acting at A and the gravity. θ is the angle $(\widehat{\vec{i}, \vec{AB}}) = (\widehat{\vec{i}, \vec{i}_S})$, x, y are the coordinate of G_1. ψ is the relative angle of S_2 with respect to S_1. We suppose that the mechanical link between the two bodies is perfect and that there is a massless motor on S_1 in oder to control ψ. The motor or drive torque about the axis $\vec{k}$ is w.*

1. *Calculate the coordinates of G_2 and of B in $\mathcal{R}$ as function of x, y, θ, ψ and of the parameter ℓ.*

2. *Is (x, y, θ, ψ) a parametrization of the motion of the system?*
3. *Calculate the velocity field V_i of S_i as a function of x, y, θ, ψ and $\dot{x}, \dot{y}, \dot{\theta}, \dot{\psi}$ for $i = 1, 2$.*
4. *Show that the inertia matrix $\mathcal{J}_G(\mathcal{B}_S)$ of S_i at G_i in $\mathcal{B}_{S_i} = (\vec{i}_{S_i}, \vec{j}_{S_i}, \vec{k})$ reads*

$$\mathcal{J}_G(\mathcal{B}_{S_i}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

5. *Calculate I. Use if necessary, $\rho = \frac{m}{2\ell}$ the linear mass density of S_i.*
6. *Calculate the kinetic wrench $\mathbb{H}_i$ of S_i for $i = 1, 2$.*
7. *Calculate the kinetic energy of the body S_i for $i = 1, 2$.*
8. *Calculate the equivalent wrenches $\mathbf{T}_{F_A}$ and $\mathbf{T}_g$ of the forces $\vec{F}_A$ and of the gravity $\vec{g} = g\vec{i}$.*
9. *Is there another external action acting on S_1?*
10. *Do the same for S_2.*
11. *Apply the first Newton law for the link $\mathcal{L}_{12}$ and the second Newton law for each body S_1 and S_2.*
12. *Compile the balance of unknowns and equations.*
13. *Find a closed system of dynamic equations. Is it possible to control ψ by u?*
14. *Find again an equivalent system of equations by use of Lagrange's equations.*

Exercice 12 *We consider a model (figure 6) of a rocket $\Sigma = \{S_1 = OA, S_2 = AB\}$ subjected to appropriate control forces such that the point O of the rocket has a constant velocity with respect to a Galilean frame $\mathcal{F}$. To model the deformations of the rocket we suppose that these deformations remain in a plane Oxy and that they are described by a relative angle θ_2 between the two pieces OA and AB of the rocket. The orientation of the system is described by the second angle θ_1. We suppose the joints are linear elastic with the same stiffness k. Each piece OA, AB is modeled by a bar (length ℓ and mass m). The engine thrust is modeled by a force $\vec{P}$ so called a follower force because it is always in the direction of the bar AB: $\vec{P} = P\frac{\vec{BA}}{||\vec{BA}||}$. $q = (\theta_1, \theta_2)$ is a Lagrange coordinate system. Gravity is neglected.*

1. *Why do we may consider that the point O is fixed for the vibration analysis? The equilibrium point is the vertical configuration namely $\theta_{1,e} = 0, \theta_{2,e} = 0$.*
2. *Calculate the Kinetic energy $C(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ of the system and deduce $\mathcal{P}^*_{acc}$ thanks to Lagrange's formula.*
3. *Calculate the virtual power of the engine thrust. Is this force deriving from a potential?*

4. What is the potential of link action $U_\ell(\theta_1, \theta_2)$? Calculate then the virtual power of link actions.

5. Write the dynamic equations?

6. Suppose $X = Col(x_1 = \theta_1 - \theta_{1,e} = \theta_1, x_2 = \theta_2 - \theta_{2,e} = \theta_2)$ is "small". Linearize the dynamic equations and deduce the mass matrix M and the stiffness matrix $K = K(P)$. Is this last matrix symmetric? What does it mean?

7. Calculate the eigenfrequencies $\omega_1 \neq \omega_2$ and the corresponding eigenmodes X_1, X_2 of vibration.

8. Investigate the divergence instability.

9. Investigate the flutter instability.

10 Figures

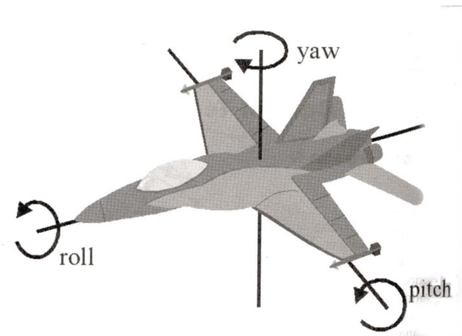

Figure 2: YawPitchRoll

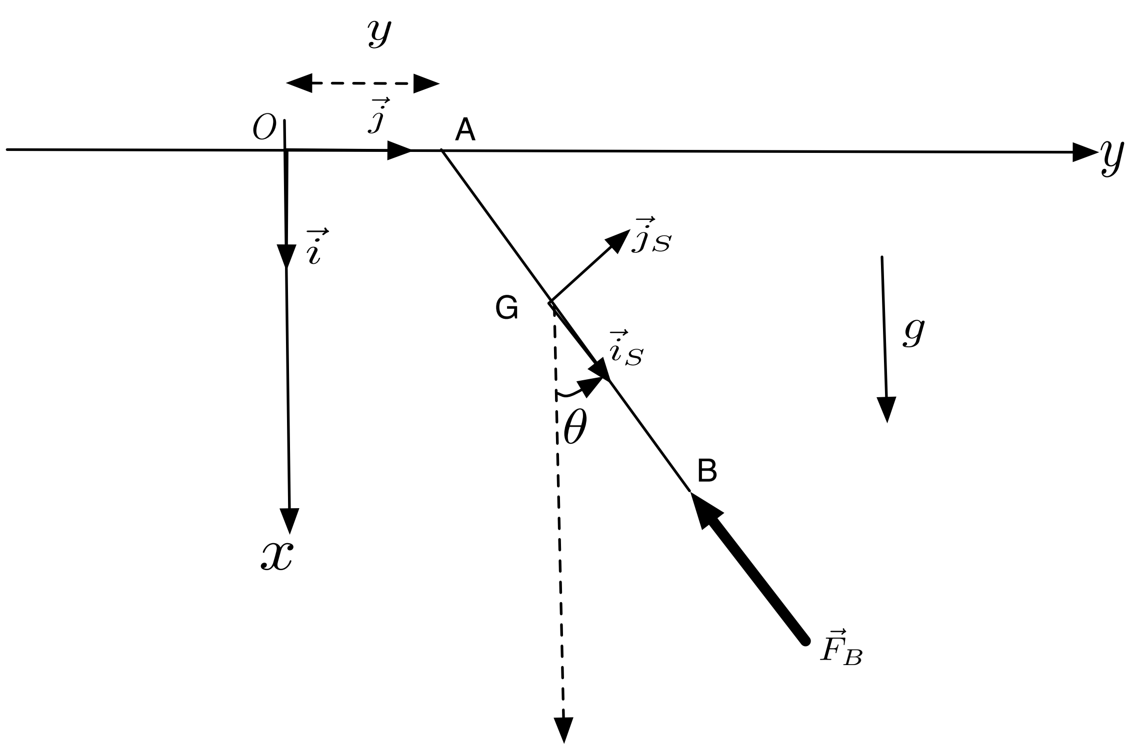

Figure 3: Body S: 1 bar

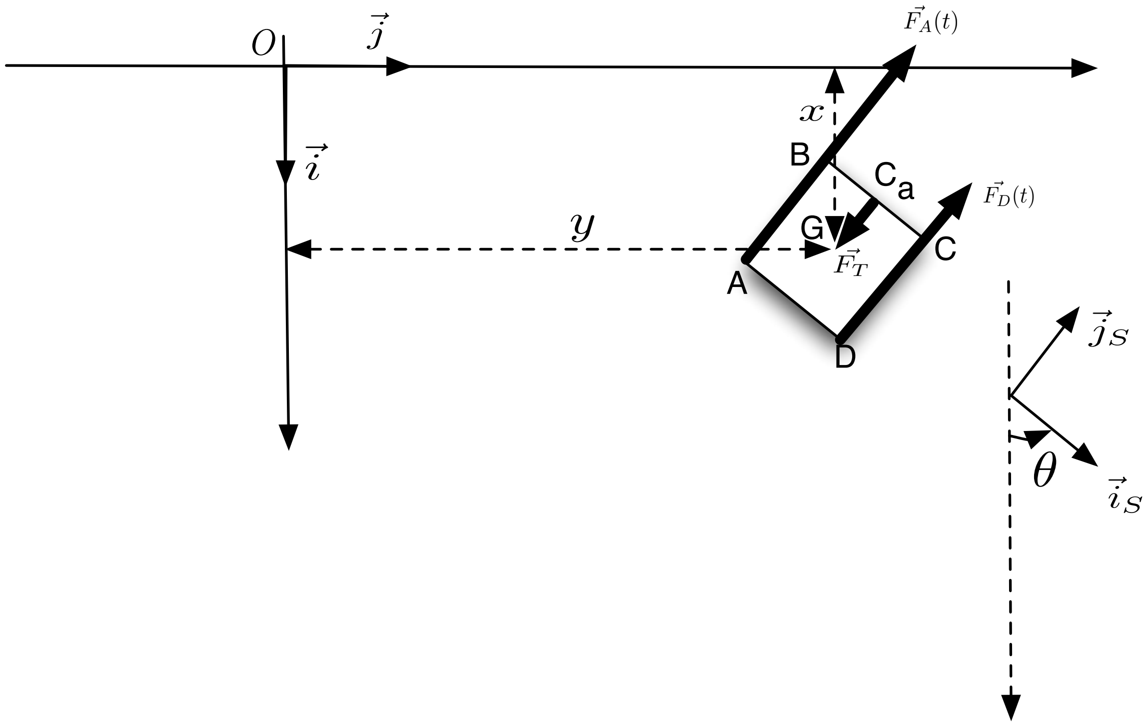

Figure 4: Body S: 1 square

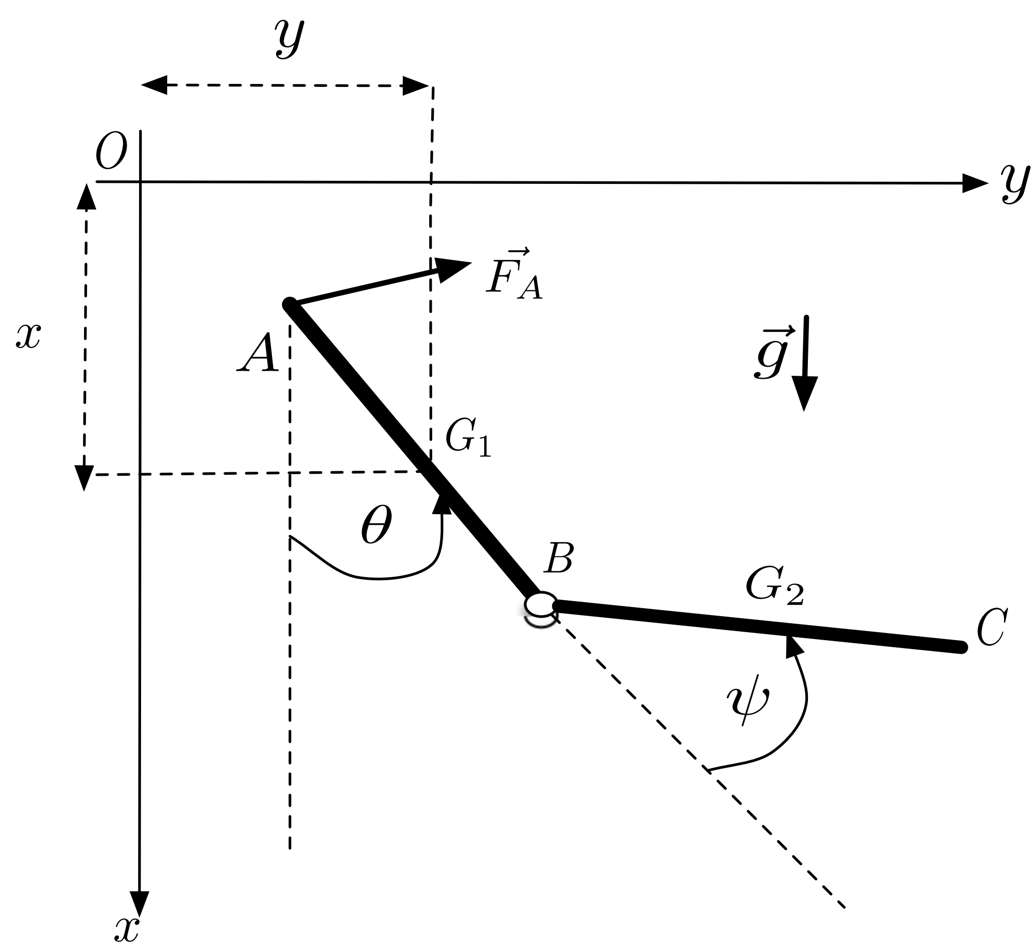

Figure 5: Control of a 2-bar system

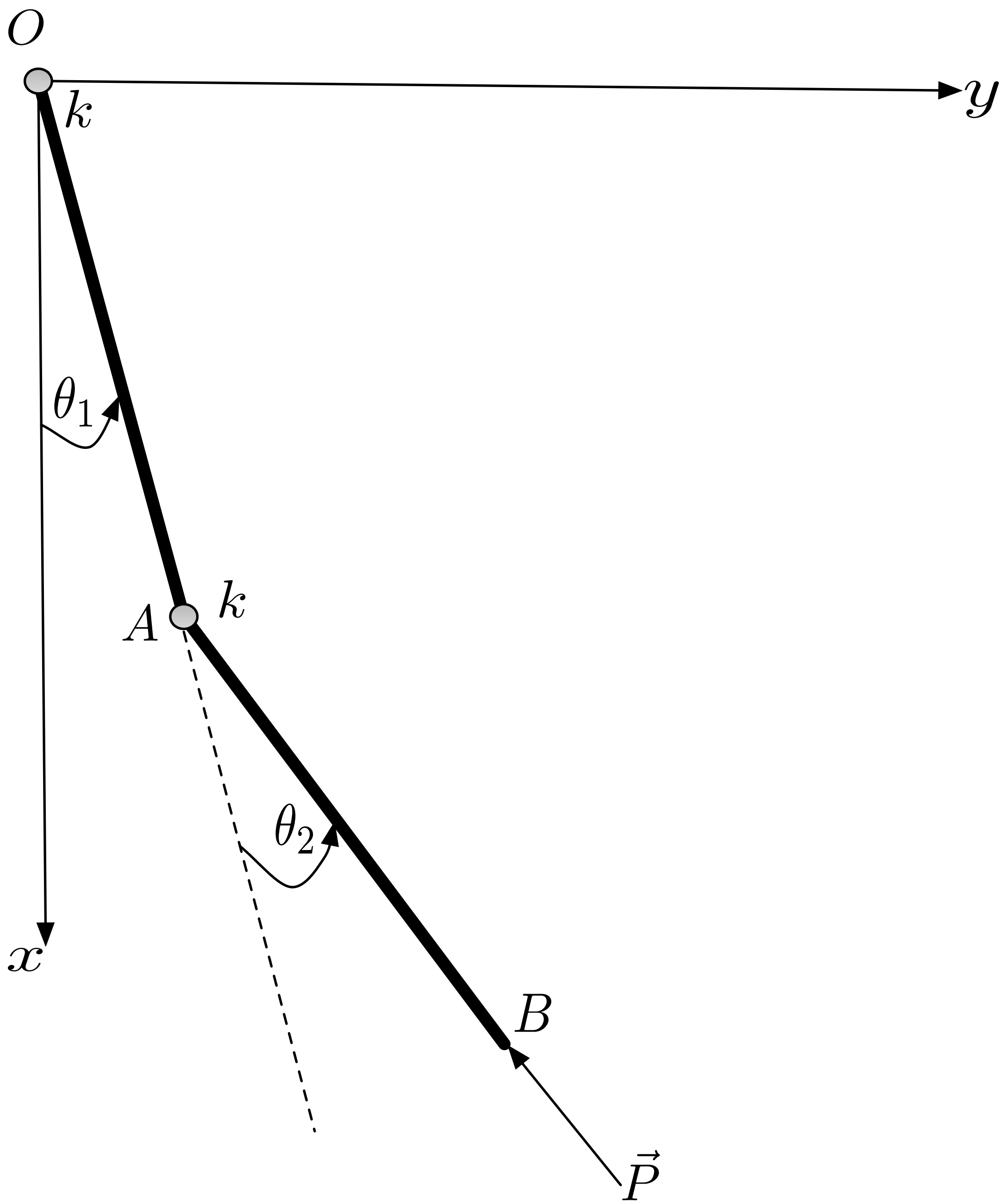

Figure 6: Thrust on a rocket. Vibration and Stability analysis